Title: **Constructive Borel-Ritt interpolation results for functions of several variables.**

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Abstract:

We give constructive proofs for two Borel-Ritt interpolation results, stating the existence of holomorphic functions on polysectors admitting an arbitrarily prescribed asymptotic expansion, whether in the sense of Gérard-Sibuya or in that of Majima. As a consequence, a new proof is obtained for the classical Borel's theorem on the existence of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ with given derivatives at $\mathbf{0}$; in fact, our construction provides functions analytic in $(\mathbb{R} - \{0\})^n$.

1 Introduction

H. Poincaré [6] gave the notion of asymptotic expansion of a function f, defined and holomorphic on an open sector S in the complex plane, as the variable tends to the vertex (without loss of generality, we will take the vertex at the origin). Borel-Ritt theorem assures the existence of holomorphic functions on a given sector S admitting an arbitrarily prescribed asymptotic expansion at 0 in S.

Outstanding generalizations of this concept to functions of several complex variables holomorphic on polysectors in $\mathbb{C}^n$ have been given by Gérard and Sibuya [2, 1979] and Majima [4, 1984]. The relation between these two concepts is now clear; the basic facts are shown in Section 3, and can be summed up as follows.

Firstly, all the derivatives of a function f, defined and holomorphic in a polysector $S \subset \mathbb{C}^n$, admit Gérard-Sibuya asymptotic expansion if, and only if, they are all bounded on bounded proper subpolysectors of S. This property suggests considering the space $\mathcal{A}(S)$ of complex functions holomorphic in S whose derivatives remain bounded on bounded proper subpolysectors of S. Actually, we will do all of our work on the space $\mathcal{A}(S, E)$ of functions defined and holomorphic on S, with values in a Fréchet space E, and verifying the same boundedness condition. On the one hand, the results and their proofs in this setting do not essentially differ from the ones for $\mathcal{A}(S)$. On the other hand, since $\mathcal{A}(S)$ is a Fréchet space when given its natural topology, it makes sense to compare $\mathcal{A}(S \times U, \mathbb{C})$ and $\mathcal{A}(S, \mathcal{A}(U, \mathbb{C}))$, where S and U are polysectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. These spaces are indeed isomorphic Fréchet spaces (Theorem 3.3).

Secondly, Majima [4, 5] gives the concept of strongly asymptotic expansion for a function in terms of a so-called total family of related functions, that needs to verify some "consistency" properties. The elements of the total family may be expressed as suitable limits of the derivatives of the function, in much the same way as the coefficients of the asymptotic expansion series in both the one variable case and the Gérard-Sibuya setting (see Proposition 3.1). These limits have a perfect sense when applied to a function in $\mathcal{A}(S,E)$, and the functions they define form a family for which consistency easily appears. This fact links both concepts of asymptotic expansion, and leads to the equivalence of:

(i) f and its derivatives admit Gérard-Sibuya asymptotic expansion at $\mathbf{0}$.

(ii) $f \in \mathcal{A}(S,E)$.

(iii) f is strongly asymptotically developable at $\mathbf{0}$.

The equivalence of both (i) and (iii) to (ii) was first obtained by the first author [3]; a proof of "(ii) if and only if (iii)" can be found in [9]. Subsequently, Zurro [12] proved that (i) and (iii) are equivalent, with techniques based on Whitney's theorem on $\mathcal{C}^\infty$ extensions.

The aim of this paper (Section 4) is to give constructive solutions for the Borel-Ritt interpolation problems relevant to these two concepts. The key for this is the isomorphism mentioned above (Theorem 3.3), since it allows to apply statements valid for vector-valued functions on sectors and induction on the number of variables.

The first problem reads as follows: given $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$, $a_{\boldsymbol{\alpha}} \in E$, does there exist $f\colon S \to E$ holomorphic and admitting this series as Gérard-Sibuya asymptotic expansion (briefly, $f \sim \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$)?

Gérard and Sibuya [2, Ch. I, Corolary 2.2.3., p. 163] showed that, fixed $q \in \mathbb{N}$, there exist f such that $D^{\boldsymbol{\beta}} f \sim \sum_{\boldsymbol{\alpha}} \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!} a_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ for every $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \le q$. Ramis [7] proved the result with no restriction on $|\boldsymbol{\beta}|$; this amounts to saying that there exists $f \in \mathcal{A}(S,E)$ with $f \sim \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$. We present an alternative proof of this fact in Theorem 4.2, resting on the construction of a solution in series form in the case of vector-valued functions defined on sectors (Proposition 4.1) and induction on the number of variables.

The corresponding Borel-Ritt problem in Majima's setting is: Given a consistent family $\mathcal{F}$, does there exist $f \in \mathcal{A}(S,E)$ such that the total family for f, denoted $TA(f)$, is precisely $\mathcal{F}$?

Majima [5, Part I, Theorem 3.1, p. 35] gave a partial solution (for complex functions): for such an $\mathcal{F}$ and for a bounded proper subpolysector T of S, there exists $f \in \mathcal{A}(T)$ such that $TA(f) = \mathcal{F}$. In order to solve the problem as initially stated, we first reformulate it in terms of the family $TA'(f)$, consisting of those elements of $TA(f)$ in $n-1$ variables; $TA'(f)$ uniquely determines $TA(f)$ and satisfies certain consistency conditions. We note that the solution given to Proposition 4.1 is also amenable to the determination (Lemma 4.3) of the behaviour of the interpolating function in the special case when the space E in which it takes its values is of the type $\mathcal{A}(U, E)$, U being a polysector. The fact that the spaces $\mathcal{A}(S \times U, E)$ and $\mathcal{A}(S, \mathcal{A}(U, E))$ are isomorphic, combined with a repeated application of Lemma 4.3, lets us apply a recurrent argument on the number of variables to obtain Theorem 4.4.

We emphasize that the technique is entirely constructive, and it strongly depends on the boundedness of the subpolysectors imposed in the definition of $\mathcal{A}(S, E)$. Subsequently, the second author and Galindo [9] have given an alternative (non-constructive) proof of this last result by a functional-analytic procedure.

As an application of Theorem 4.2, we give a new proof of the well-known Borel's theorem on the existence of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ with given derivatives at $\mathbf{0}$, with the special feature that solutions are indeed analytic in $(\mathbb{R} - \{0\})^n$.

We would like to mention that the two concepts of asymptotic expansion in several variables dealt with in this paper are by no means the only ones occurring in the literature. In Section 5 we comment on the definition considered by Gérard and Jurkat [1], who have obtained an extension of the preparation theorem and the division theorem to a fairly general asymptotic case; and on the concept of uniform asymptotic expansions treated by Wasow [11] and Gérard and Sibuya [2], with applications to the study of ordinary differential equations with parameters and completely integrable pfaffian systems, respectively.

2 Notation

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $n \in \mathbb{N}$, $n \geq 1$, put $N = \{1, 2, \ldots, n\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$ be two multiindices, $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, and m a natural number. We set

$$m\boldsymbol{\alpha} = (m\alpha_1, m\alpha_2, \ldots, m\alpha_n), \qquad \boldsymbol{\alpha}+\boldsymbol{\beta} = (\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots, \alpha_n+\beta_n),$$
$$|\boldsymbol{\alpha}| = \alpha_1+\alpha_2+\cdots+\alpha_n, \qquad \boldsymbol{\alpha}! = \alpha_1!\,\alpha_2!\cdots\alpha_n!,$$
$$\boldsymbol{\alpha} \leq \boldsymbol{\beta} \Leftrightarrow \alpha_j \leq \beta_j,\ j \in N, \qquad \boldsymbol{\alpha} < \boldsymbol{\beta} \Leftrightarrow \alpha_j < \beta_j,\ j \in N,$$
$$\mathbf{1} = (1,1,\ldots,1), \qquad \boldsymbol{e}_j = (0,\ldots,0,\overset{j)}{1},0,\ldots,0),$$
$$\boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n}, \qquad |\boldsymbol{z}^{\boldsymbol{\alpha}}| = |\boldsymbol{z}|^{\boldsymbol{\alpha}} = |z_1|^{\alpha_1}|z_2|^{\alpha_2}\cdots|z_n|^{\alpha_n},$$
$$\|\boldsymbol{z}\| = |z_1|+|z_2|+\cdots+|z_n|, \qquad D^{\boldsymbol{\alpha}} = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_n^{\alpha_n}}.$$

If J is a nonempty subset of N, the number of elements of J will be $\#J$.

For $j = 1,2,\ldots,n$, consider an open sector in $\mathbb{C}$ with vertex at the origin,

$$S_j = \{z \in \mathbb{C} \colon \theta_{1j} < \arg(z) < \theta_{2j},\ 0 < |z| < R_j\},$$

where $0 < \theta_{2j} - \theta_{1j} \leq 2\pi$ and $R_j \in (0,+\infty]$.

A cartesian product of open sectors in $\mathbb{C}$ with vertex at 0, $S = \prod_{j=1}^{n} S_j \subset \mathbb{C}^n$, will be called (open) *polysector* in $\mathbb{C}^n$ with vertex at $\mathbf{0}$.

We say a polysector $T = \prod_{j=1}^{n} T_j$ in $\mathbb{C}^n$ (with vertex at the origin) is a *bounded proper subpolysector* of S if it is bounded and $\overline{T_j} \subset S_j \cup \{0\}$, $j = 1,2,\ldots,n$. In this case, we write $T \ll S$.

If $J = \{j_1 < j_2 < \ldots < j_k\}$ is a nonempty subset of N and $\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n$, we put $\boldsymbol{z}_J = (z_{j_1}, z_{j_2}, \ldots, z_{j_k})$. Let J and L be nonempty disjoint subsets of N. For $\boldsymbol{z}_J \in \mathbb{C}^J$ and $\boldsymbol{z}_L \in \mathbb{C}^L$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)$ represents the element of $\mathbb{C}^{J\cup L}$ satisfying $(\boldsymbol{z}_J,\boldsymbol{z}_L)_J = \boldsymbol{z}_J$, $(\boldsymbol{z}_J,\boldsymbol{z}_L)_L = \boldsymbol{z}_L$; we also write $J^c = N - J$, and for $j \in N$ we use j^c instead of $\{j\}^c$. In particular, we shall use these conventions for multiindices.

If $S = \prod_{j=1}^{n} S_j$ is a polysector of $\mathbb{C}^n$, then $S_J = \prod_{j\in J} S_j \subset \mathbb{C}^J$.

For $\boldsymbol{\delta} \in (0,\infty)^n$, we denote by $D_{\boldsymbol{\delta}}(\boldsymbol{z})$ (respectively, $\overline{D}_{\boldsymbol{\delta}}(\boldsymbol{z})$) the open (resp. closed) polydisc centered at $\boldsymbol{z}$ with polyradius $\boldsymbol{\delta}$; $\partial_0 D_{\boldsymbol{\delta}}(\boldsymbol{z})$ will be their distinguished boundary. In case $\boldsymbol{\delta} = (\delta,\delta,\ldots,\delta)$, we simply write $D_\delta(\boldsymbol{z})$, and so on.

3 Asymptotic expansions for functions on polysectors

We generalize the notion of asymptotic expansion given by Gérard and Sibuya [2] to functions valued in a Fréchet space E.

Let $S \subset \mathbb{C}^n$ be a polysector with vertex at $\mathbf{0}$, and $f \colon S \to E$ be holomorphic. We say f admits *asymptotic expansion in the sense of Gérard and Sibuya* (GS-ae, for short) at $\mathbf{0}$ if there exists a (formal) power series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$, with $a_{\boldsymbol{\alpha}} \in E$, such that for every continuous seminorm (briefly, cs) p on E, $T \ll S$ and $m \in \mathbb{N}$, there exists a constant $C > 0$ (depending on p, T and m) verifying that

$$p\left(f(\boldsymbol{z}) - \sum_{j=0}^{m}\sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}\right) \leq C\|\boldsymbol{z}\|^{m+1}, \qquad \boldsymbol{z} \in T.$$

In this situation, we write

$$f \sim_S \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}},\ \boldsymbol{z} \to \mathbf{0},$$

or simply $f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$ when there is no ambiguity.

The next proposition determines the coefficients in $\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$ in terms of limits at $\mathbf{0}$, following suitable subsets of S, of the derivatives of f.

Proposition 3.1 *If $f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$, then for every compact $K \subset S$ and $\boldsymbol{\alpha} \in \mathbb{N}^n$,*

$$\lim_{\boldsymbol{z}\to\mathbf{0},\, \boldsymbol{z}\in\widetilde{K}} \frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{z})}{\boldsymbol{\alpha}!} = a_{\boldsymbol{\alpha}},$$

where $\widetilde{K} = \{\, \lambda\boldsymbol{z}\colon \lambda \in (0,1],\ \boldsymbol{z} \in K\,\}$.

Proof: Put $|\boldsymbol{\alpha}| = m$ and $g(\boldsymbol{z}) = \sum_{j=0}^{m}\sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$, $\boldsymbol{z} \in S$. Consider $T \ll S$ such that $K \subset T$. If $d = \mathrm{dist}(K, \mathbb{C}^n - T)$ and $M = \max\{\|\boldsymbol{z}\|\colon \boldsymbol{z} \in K\}$, take $r = \frac{d}{2Mn}$; it is easy to check that for every $\boldsymbol{z} \in K$ we have $\overline{D}_{r\|\boldsymbol{z}\|}(\boldsymbol{z}) \subset T$.

Given a cs p on E and $\varepsilon > 0$, there exists $\rho > 0$ such that

$$p\left(\frac{f(\boldsymbol{\omega}) - g(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^m}\right) < \varepsilon, \qquad \boldsymbol{\omega} \in \overline{D}_\rho(\mathbf{0}) \cap T.$$

If we choose $\delta = \frac{\rho}{n(1+nr)}$, then $\overline{D}_{r\|\boldsymbol{z}\|}(\boldsymbol{z}) \subset \overline{D}_\rho(\mathbf{0}) \cap T$ whenever $\boldsymbol{z} \in D_\delta(\mathbf{0}) \cap \widetilde{K}$. So, for such $\boldsymbol{z}$ we get

$$\frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{z})}{\boldsymbol{\alpha}!} - a_{\boldsymbol{\alpha}} = \frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{z}) - D^{\boldsymbol{\alpha}} g(\boldsymbol{z})}{\boldsymbol{\alpha}!} = \frac{1}{(2\pi i)^n}\int_{\partial_0 D_{r\|\boldsymbol{z}\|}(\boldsymbol{z})} \frac{f(\boldsymbol{\omega}) - g(\boldsymbol{\omega})}{(\boldsymbol{\omega} - \boldsymbol{z})^{\boldsymbol{\alpha}+\mathbf{1}}}\, d\boldsymbol{\omega},$$

and hence

$$\begin{aligned} p\left(\frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{z})}{\boldsymbol{\alpha}!} - a_{\boldsymbol{\alpha}}\right) &\leq \frac{(2\pi r\|\boldsymbol{z}\|)^n}{(2\pi)^n (r\|\boldsymbol{z}\|)^{m+n}} \sup_{\boldsymbol{\omega}\in\partial_0 D_{r\|\boldsymbol{z}\|}(\boldsymbol{z})} p(f(\boldsymbol{\omega}) - g(\boldsymbol{\omega})) \\ &\leq \frac{1}{(r\|\boldsymbol{z}\|)^m} \sup_{\boldsymbol{\omega}\in\partial_0 D_{r\|\boldsymbol{z}\|}(\boldsymbol{z})} p\left(\frac{f(\boldsymbol{\omega}) - g(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^m}\right)\|\boldsymbol{\omega}\|^m < \frac{(1+nr)^m}{r^m}\varepsilon, \end{aligned}$$

what concludes the proof. □

As an easy consequence we get that if $f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ and for some $\boldsymbol{\beta} \in \mathbb{N}^n$, $D^{\boldsymbol{\beta}} f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} b_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$, then $b_{\boldsymbol{\alpha}} = \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!} a_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$, $\boldsymbol{\alpha} \in \mathbb{N}^n$.

Also, if f is such that $D^{\boldsymbol{\beta}} f$ admits GS-ae for every $\boldsymbol{\beta} \in \mathbb{N}^n$, then $D^{\boldsymbol{\beta}} f$ is bounded on every $T \ll S$. And conversely, if the last statement holds, derivatives are lipschitzian and Taylor's formula implies that they admit GS-ae [3, 12]. So, we are naturally led to bring in the complex vector space $\mathcal{A}(S, E)$ of holomorphic functions $f : S \to E$ such that for every cs p on E, $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $T \ll S$ we have

$$Q_{p,\boldsymbol{\alpha},T}(f) = \sup\{\, p(D^{\boldsymbol{\alpha}} f(\boldsymbol{z})) : \boldsymbol{z} \in T \,\} < \infty.$$

Theorem 3.2 *$f \in \mathcal{A}(S, E)$ if, and only if, $D^{\boldsymbol{\alpha}} f$ admits GS-ae for every $\boldsymbol{\alpha}$.*

It is evident that $\mathcal{A}(S, E)$ is a differential algebra. Equip $\mathcal{A}(S, E)$ with the topology generated by the family $\{Q_{p,\boldsymbol{\alpha},T}\}$ of seminorms. This makes $\mathcal{A}(S, E)$ a Fréchet space.

Next, we focus on the relations between the spaces $\mathcal{A}(S \times U, E)$ and $\mathcal{A}(S, \mathcal{A}(U, E))$, where $S \subset \mathbb{C}^n$ and $U \subset \mathbb{C}^m$ are polysectors. The somewhat technical proof of the following basic result is omitted.

Theorem 3.3 (i) *The map*

$$f \in \mathcal{A}(S \times U, E) \to f^* \in \mathcal{A}(S, \mathcal{A}(U, E)),$$

defined for every $\boldsymbol{z} \in S$ by $f^*(\boldsymbol{z}) = f(\boldsymbol{z}, \cdot)$, is a topological isomorphism, and for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $\boldsymbol{\beta} \in \mathbb{N}^m$ we have

$$D^{(\boldsymbol{\alpha},\boldsymbol{\beta})} f(\boldsymbol{z}, \boldsymbol{\omega}) = D^{\boldsymbol{\beta}}(D^{\boldsymbol{\alpha}} f^*(\boldsymbol{z}))(\boldsymbol{\omega}), \qquad (\boldsymbol{z}, \boldsymbol{\omega}) \in S \times U.$$

(ii) *Let $f \in \mathcal{A}(S \times U, E)$. For $\boldsymbol{\omega} \in U$, $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $T \ll S$,*

$$\lim_{\substack{\boldsymbol{z}\to\boldsymbol{0}\\ \boldsymbol{z}\in T}} \frac{D^{(\boldsymbol{\alpha},\boldsymbol{0})} f(\boldsymbol{z}, \boldsymbol{\omega})}{\boldsymbol{\alpha}!} = f_{(\boldsymbol{\alpha},\boldsymbol{0})}(\boldsymbol{\omega})$$

exists, it is uniform on every $V \ll U$, and it defines $f_{(\boldsymbol{\alpha},\boldsymbol{0})} \in \mathcal{A}(U, E)$.

(iii) *For every $\boldsymbol{\gamma} \in \mathbb{N}^m$, $(D^{(\boldsymbol{0},\boldsymbol{\gamma})} f)^* \sim_S \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} (D^{\boldsymbol{\gamma}} f_{(\boldsymbol{\alpha},\boldsymbol{0})}) \boldsymbol{z}^{\boldsymbol{\alpha}}$.*

Let us recall now how the link is established between the space $\mathcal{A}(S, E)$ and that of strongly asymptotically developable functions in the sense of Majima [4, 5]. These results come from [3], and some proofs can be found in [9].

Let $f \in \mathcal{A}(S,E)$. According to Theorem 3.3, if $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, we can define a function from S_{J^c} to E by

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_J\to\mathbf{0}\\ \boldsymbol{z}_J\in T_J}} \frac{D^{(\boldsymbol{\alpha}_J,\mathbf{0}_{J^c})}f(\boldsymbol{z})}{\boldsymbol{\alpha}_J!}, \quad \boldsymbol{z}_{J^c} \in S_{J^c}, \tag{3.1}$$

for any subpolysector T_J of S_J; $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c},E)$ (setting $\mathcal{A}(S_{N^c},E)=E$).

In this way, we may associate with f a family

$$\mathcal{F}(f) = \{\, f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c},E)\colon \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \,\}.$$

Proposition 3.4 (Consistency conditions) *Let $f \in \mathcal{A}(S,E)$. Then, for every disjoint nonempty subsets J and L of N, for $\boldsymbol{\alpha}_J \in \mathbb{N}^J$ and $\boldsymbol{\alpha}_L \in \mathbb{N}^L$, and for $T_L \ll S_L$,*

$$\lim_{\substack{\boldsymbol{z}_L\to\mathbf{0}\\ \boldsymbol{z}_L\in T_L}} \frac{D^{(\boldsymbol{\alpha}_L,\mathbf{0}_{(J\cup L)^c})}f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_L!} = f_{(\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)^c}); \tag{3.2}$$

the limit is uniform on bounded proper subpolysectors of $S_{(J\cup L)^c}$ whenever $J \cup L \neq N$.

Hereafter, we will say that a family

$$\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c},E)\colon \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \,\},$$

or briefly $\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \,\}$, is *consistent* if it verifies (3.2).

We then adapt to this context the notion of asymptotic expansion given by Majima.

A holomorphic function $f\colon S \to E$ is said to be *strongly asymptotically developable* at $\mathbf{0}$ if there exists a family

$$\mathcal{F} = \left\{\, f_{\boldsymbol{\alpha}_J}\colon \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \,\right\},$$

where $f_{\boldsymbol{\alpha}_J}$ is a holomorphic function from S_{J^c} to E when $J \neq N$, and $f_{\boldsymbol{\alpha}_J} \in E$ when $J = N$, such that the following holds: if we define

$$App_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z}) = \sum_{\emptyset\neq J\subset N} (-1)^{\#J+1} \sum_{\substack{\boldsymbol{\beta}_J\in\mathbb{N}^J\\ \boldsymbol{\beta}_J\le\boldsymbol{\alpha}_J-\mathbf{1}_J}} f_{\boldsymbol{\beta}_J}(\boldsymbol{z}_{J^c})\boldsymbol{z}_J^{\boldsymbol{\beta}_J}, \quad \boldsymbol{\alpha} \in \mathbb{N}^n,\ \boldsymbol{z} \in S,$$

then for every cs p on E, $T \ll S$ and $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$\sup\left\{ p\Big(\frac{f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big)\colon \boldsymbol{z} \in T \right\} < \infty.$$

Under these conditions, $\mathcal{F}$ will be called *total family of strongly asymptotic expansion* associated with f, and will be denoted by $TA(f)$. For $\boldsymbol{\alpha} \in \mathbb{N}^n$ the holomorphic function $App_{\boldsymbol{\alpha}}(\mathcal{F})\colon S \to E$ is called the *approximate function* of order $\boldsymbol{\alpha}$ corresponding to the family $\mathcal{F}$.

Theorem 3.5 *Let $f: S \to E$ be holomorphic. Then f is strongly asymptotically developable at $\mathbf{0}$ in S if, and only if, $f \in \mathcal{A}(S,E)$. If this is the case, then $\mathcal{F}(f) = TA(f)$.*

By (3.1), $TA(f)$ is unique. So, the approximate functions may be denoted as $App_{\boldsymbol{\alpha}}(f)$, $\boldsymbol{\alpha} \in \mathbb{N}^n$. The elements of $TA(f)$ are strongly asymptotically developable, and from the consistency conditions we see that

$$TA(f_{\boldsymbol{\alpha}_J}) = \{\, f_{(\boldsymbol{\alpha}_J,\boldsymbol{\beta}_L)}: \emptyset \neq L \subset J^c,\ \boldsymbol{\beta}_L \in \mathbb{N}^L \,\}.$$

We note that the notion of consistent family was given by Majima (see [5, Part I, p. 25]), though we have arrived at it from a different setting. Also, it is evident that the family $\{P_{p,T,\boldsymbol{\alpha}}\}$ of seminorms, defined on $\mathcal{A}(S,E)$ for every cs p on E, $T \ll S$ and $\boldsymbol{\alpha} \in \mathbb{N}^n$ as

$$P_{p,T,\boldsymbol{\alpha}}(f) = \sup \left\{ p\left(\frac{f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}} \right) : \boldsymbol{z} \in T \right\},$$

generates in $\mathcal{A}(S,E)$ the original topology.

4 Interpolation results

We begin giving Borel-Ritt theorem for vector-valued functions of one variable; the proof follows that for the classical result, with the suitable modifications.

Proposition 4.1 *Let S be a sector and $\{a_n\}_{n=0}^{\infty}$ a sequence in E. Then there exists $f \in \mathcal{A}(S,E)$ such that $f \sim \sum_{n=0}^{\infty} a_n z^n$.*

Proof: We may suppose, without loss of generality, that

$$S = \{z \in \mathbb{C}: -\pi < \arg(z) < \pi\}.$$

Let $\{p_m\}_{m=0}^{\infty}$ be an increasing sequence of seminorms generating the topology of E, and define $b_m = \frac{1}{m!}$, or $b_m = \frac{1}{m!\,p_m(a_m)}$, according to whether $p_m(a_m) = 0$ or not. Consider the function $h_m(z) = 1 - \exp(-b_m/\sqrt{z})$, holomorphic in S; in [8, Ch.9, §6.4] it is proved that

$$|h_m(z)| \leq b_m |z|^{-1/2}, \quad z \in S, \tag{4.1}$$

and

$$\lim_{\substack{z\to 0 \\ z\in T\ll S}} |1 - h_m(z)||z|^{-k} = 0, \quad k = 0, 1, \ldots. \tag{4.2}$$

From (4.1) it follows that the series $\sum_{m=0}^{\infty} a_m h_m(z) z^m$ converges uniformly on compact subsets of S, so defining a holomorphic function $f\colon S \to E$. It remains to show that for every bounded proper subsector T of S and $m \in \mathbb{N}$ we have

$$\lim_{\substack{z\to 0\\ z\in T}} \frac{f(z)-\sum_{k=0}^{m} a_k z^k}{z^m} = 0.$$

Fix T and m, and take $n > m$. For $z \in T$ we may write

$$p_n\Big(\frac{f(z)-\sum_{k=0}^{m} a_k z^k}{z^m}\Big) \le \sum_{k=0}^{m} p_n(a_k)|h_k(z)-1||z|^{k-m}$$
$$+ \sum_{k=m+1}^{n} p_n(a_k) b_k |z|^{k-m-1/2} + \sum_{k=n+1}^{\infty} p_n(a_k)\frac{b_k}{|z|^{1/2}}|z|^{k-m}.$$

By (4.2), the first sum tends to 0 when z does; the same is obviously valid for the second sum. Finally, if $|z| < 1$,

$$\sum_{k=n+1}^{\infty} p_n(a_k)\frac{b_k}{|z|^{1/2}}|z|^{k-m} \le \frac{1}{|z|^{1/2}}\sum_{k=n+1}^{\infty} p_k(a_k) b_k |z|^{k-m} \le \frac{1}{|z|^{1/2}}\sum_{k=n+1}^{\infty}\frac{|z|^{k-m}}{k!},$$

what concludes the proof. $\square$

We can already prove a Borel-Ritt type result for functions admitting a given GS-ae on polysectors, up to any order of differentiation.

Theorem 4.2 *Let $S \subset \mathbb{C}^n$ be a polysector and $\{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ be a multisequence in E. Then, there exists $f \in \mathcal{A}(S,E)$ such that $f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$.*

Proof: We proceed by induction on n. For $n = 1$ this is the previous result. Suppose the result holds for $n-1$ ($n \ge 2$). We write an element in S as

$$(\omega, \boldsymbol{z}) \in S_1 \times S_{1^c} = S_1 \times U = S.$$

By the induction hypothesis, for each $m \in \mathbb{N}$ there exists $\varphi_m \in \mathcal{A}(U,E)$ such that $\varphi_m \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^{n-1}} a_{(m,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}$. Proposition 4.1 assures there exists $f^* \in \mathcal{A}(S_1, \mathcal{A}(U,E))$ such that $f^* \sim \sum_{m=0}^{\infty} \varphi_m \omega^m$. The isomorphism in Theorem 3.3 provides the solution $f \in \mathcal{A}(S,E)$, given by $f(\omega,\boldsymbol{z}) = f^*(\omega)(\boldsymbol{z})$. Indeed, let p be a cs on E and T a bounded proper subpolysector of S, $T = T_1 \times \prod_{j=2}^{n} T_j = T_1 \times V$. There exist $C_m > 0$, $K_m > 0$ such that

$$p\Big(\varphi_k(\boldsymbol{z}) - \sum_{|\boldsymbol{\alpha}|\le m-k} a_{(k,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}\Big) \le C_m\|\boldsymbol{z}\|^{m-k+1}, \quad \boldsymbol{z}\in V,\ k = 0,1,\ldots,m,$$

and

$$p\Big(f(\omega,\boldsymbol{z})-\sum_{k=0}^{m}\varphi_k(\boldsymbol{z})\omega^k\Big)\le K_m|\omega|^{m+1},\quad \omega\in T_1,\ \boldsymbol{z}\in V.$$

Hence, for $(\omega,\boldsymbol{z})\in T$ we have

$$\begin{aligned}
&p\Big(f(\omega,\boldsymbol{z})-\sum_{|(k,\boldsymbol{\alpha})|\le m}a_{(k,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}\omega^k\Big)\\
&\le\ p\Big(f(\omega,\boldsymbol{z})-\sum_{k=0}^{m}\varphi_k(\boldsymbol{z})\omega^k\Big)+p\Big(\sum_{k=0}^{m}\varphi_k(\boldsymbol{z})\omega^k-\sum_{|(k,\boldsymbol{\alpha})|\le m}a_{(k,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}\omega^k\Big)\\
&\le\ K_m|\omega|^{m+1}+p\Big(\sum_{k=0}^{m}\varphi_k(\boldsymbol{z})\omega^k-\sum_{k=0}^{m}\sum_{|\boldsymbol{\alpha}|\le m-k}a_{(k,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}\omega^k\Big)\\
&\le\ K_m|\omega|^{m+1}+\sum_{k=0}^{m}p\Big(\varphi_k(\boldsymbol{z})-\sum_{|\boldsymbol{\alpha}|\le m-k}a_{(k,\boldsymbol{\alpha})}\boldsymbol{z}^{\boldsymbol{\alpha}}\Big)|\omega|^k\\
&\le\ K_m|\omega|^{m+1}+\sum_{k=0}^{m}C_m\|\boldsymbol{z}\|^{m-k+1}|\omega|^k\le\max\{K_m,C_m\}(|\omega|+\|\boldsymbol{z}\|)^{m+1},
\end{aligned}$$

as desired. □

The next result provides additional information on the interpolating function constructed in Proposition 4.1 when it takes its values in a Fréchet space of the type $\mathcal{A}(S,E)$. It will be useful on solving Borel-Ritt problem in Majima's setting.

Lemma 4.3 *Let U be a sector in $\mathbb{C}$ and $S=\prod_{j=1}^{n}S_j$ a polysector in $\mathbb{C}^n$. Suppose that for a sequence $\{f_m\}_{m=0}^{\infty}$ in $\mathcal{A}(S,E)$ there exists a nonempty $J\subset N=\{1,2,\ldots,n\}$ such that for every $j\in J$, $T_j\ll S_j$ and $q\in\mathbb{N}$ we have*

$$\lim_{\substack{z_j\to 0\\ z_j\in T_j}} D^{qe_j}f_m(\boldsymbol{z})=0,\qquad m=0,1,2,\ldots,$$

uniformly on bounded proper subpolysectors of S_{j^c}.

Then, there exists a function $f^*\in\mathcal{A}(U,\mathcal{A}(S,E))$ with $f^*\sim\sum_{m=0}^{\infty}f_m\omega^m$, and such that f (given by the isomorphism) verifies that for every $j\in J$, $T_j\ll S_j$ and $q\in\mathbb{N}$,

$$\lim_{\substack{z_j\to 0\\ z_j\in T_j}} D^{(0,qe_j)}f(\omega,\boldsymbol{z})=0$$

uniformly on every $V\times T_{j^c}$, where $V\ll U$ and $T_{j^c}\ll S_{j^c}$.

Proof: It suffices to work with $U=\{\omega\in\mathbb{C}\colon|\arg(\omega)|<\pi\}$. Consider an increasing sequence of continuous seminorms $\{p_m\}_{m=0}^{\infty}$ defining the topology of E, and an exhausting sequence $\{H_m\}_{m=0}^{\infty}$ of bounded proper subpolysectors of S. It is clear that $\{P_m\}_{m=0}^{\infty}$, given by

$$P_m(f) = \sup_{|\boldsymbol{\alpha}|\le m} \sup_{\boldsymbol{z}\in H_m} p_m(D^{\boldsymbol{\alpha}} f(\boldsymbol{z})), \qquad f \in \mathcal{A}(S,E),$$

is an increasing sequence of continuous seminorms in $\mathcal{A}(S,E)$ defining its natural topology. Now, Proposition 4.1 allows to construct a function $f^* \in \mathcal{A}(U, \mathcal{A}(S,E))$, given by

$$f^*(\omega) = \sum_{m=0}^{\infty} f_m h_m(\omega)\omega^m, \qquad \omega \in U,$$

such that $f^* \sim \sum_{m=0}^{\infty} f_m \omega^m$. By Theorem 3.3, the function $f\colon U\times S \to E$, $f(\omega,\boldsymbol{z}) = f^*(\omega)(\boldsymbol{z})$, belongs to $\mathcal{A}(U\times S, E)$. Let us prove that for every $\boldsymbol{\alpha}\in\mathbb{N}^n$,

$$\sum_{m=0}^{\infty} D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z}) h_m(\omega)\omega^m$$

converges (to $D^{(0,\boldsymbol{\alpha})} f(\omega,\boldsymbol{z})$) normally on the sets $V\times T$, where $V \ll U$ and $T \ll S$. Indeed, take $m_1\in\mathbb{N}$ such that $T \ll H_{m_1}$ and $|\boldsymbol{\alpha}| \le m_1$. Put $\mu = \sup_{\omega\in V}|\omega|$; then, for $\nu\in\mathbb{N}$ and every $m \ge \max\{m_1,\nu,1\}$ we have

$$\begin{aligned} p_\nu(D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z}) h_m(\omega)\omega^m) &\le p_m(D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z}) h_m(\omega)\omega^m) \\ &\le |1-\exp(\frac{-b_m}{\sqrt{\omega}})|\, p_m(D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z}))\, |\omega|^m \\ &\le b_m\, P_m(f_m)\, |\omega|^m \le \frac{1}{m!}\,\mu^{m-1/2}. \end{aligned}$$

On the other hand, since $\lim_{\omega\to 0,\,\omega\in V}(1-\exp(\frac{-b_0}{\sqrt{\omega}})) = 1$, we have that

$$\sup_{(\omega,\boldsymbol{z})\in V\times T} |1-\exp(\frac{-b_0}{\sqrt{\omega}})|\, p_\nu(D^{\boldsymbol{\alpha}} f_0(\boldsymbol{z})) = M_0 < +\infty,$$

and hence

$$\begin{aligned} &\sum_{m=0}^{\infty} p_\nu(D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z}) h_m(\omega)\omega^m) \\ &\le\ M_0 + \sum_{m=1}^{m_0} b_m \mu^{m-1/2} \sup_{\boldsymbol{z}\in T} p_\nu(D^{\boldsymbol{\alpha}} f_m(\boldsymbol{z})) + \sum_{m=m_0+1}^{\infty} \frac{1}{m!}\mu^{m-1/2} < +\infty, \end{aligned}$$

as desired. Taking this fact into account, the proof will be complete if we show that for $j\in J$, $T_j \ll S_j$ and $q\in\mathbb{N}$,

$$\lim_{\substack{z_j\to 0\\ z_j\in T_j}} \sum_{m=0}^{\infty} D^{q e_j} f_m(\boldsymbol{z}) h_m(\omega)\omega^m = 0$$

uniformly on $V \times T_{j^c}$, where $V \ll U$ and $T_{j^c} \ll S_{j^c}$. Given $\varepsilon > 0$ and a cs p on E, there exists $m_0 \geq 1$ such that

$$p(\sum_{m=m_0+1}^{\infty} D^{qe_j} f_m(\boldsymbol{z}) h_m(\omega)\omega^m) < \frac{\varepsilon}{4}, \quad (\omega, \boldsymbol{z}) \in V \times T_j \times T_{j^c}.$$

Put $\mu = \sup_{\omega \in V} |\omega|$ and $M = \sup_{1 \leq m \leq m_0} b_m \mu^{m-1/2}$; by hypothesis, there exists $\delta > 0$ such that for every $\boldsymbol{z} = (z_j, \boldsymbol{z}_{j^c}) \in (T_j \cap D(0,\delta)) \times T_{j^c}$ we have

$$p(D^{qe_j} f_m(\boldsymbol{z})) < \frac{\varepsilon}{2Mm_0}, \quad 1 \leq m \leq m_0,$$

and

$$|1 - \exp(\frac{-b_0}{\sqrt{\omega}})| p(D^{qe_j} f_0(\boldsymbol{z})) < \frac{\varepsilon}{4}, \qquad \omega \in V.$$

So, for every $\boldsymbol{z} = (z_j, \boldsymbol{z}_{j^c}) \in (T_j \cap D(0,\delta)) \times T_{j^c}$ and every $\omega \in V$,

$$\begin{aligned} & p(\sum_{m=0}^{\infty} D^{qe_j} f_m(\boldsymbol{z}) h_m(\omega)\omega^m) \\ & \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} \frac{\varepsilon}{2Mm_0} b_m \mu^{m-1/2} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

and the conclusion follows. □

The following Borel-Ritt problem may be posed:

Given a consistent family $\mathcal{F}$, prove the existence of a function $f \in \mathcal{A}(S,E)$ such that $TA(f) = \mathcal{F}$.

We now give the framework needed to solve it. For $n > 1$ and $f \in \mathcal{A}(S,E)$ we call

$$TA'(f) = \{ f_{m_{\{j\}}} \in \mathcal{A}(S_{j^c}, E) : j \in N, \ m \in \mathbb{N} \}$$

the *first order family* associated to f. It consists of those elements of $TA(f)$ in $n-1$ variables. For convenience, we write f_{jm} instead of $f_{m_{\{j\}}}$. $TA(f)$ is consistent, hence $TA'(f)$ satisfies the following *first order consistency conditions*:

For every $L \subset N$ consisting of at least two elements, every $\boldsymbol{\alpha}_L \in \mathbb{N}^L$ and every $j, \ell \in L$, and $T \ll S$, we have

$$\lim_{\substack{\boldsymbol{z}_{L-\{j\}} \to 0 \\ \boldsymbol{z}_{L-\{j\}} \in T_{L-\{j\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{j\}}, \boldsymbol{0}_{L^c})} f_{j\alpha_j}(\boldsymbol{z}_{j^c})}{\boldsymbol{\alpha}_{L-\{j\}}!} = \lim_{\substack{\boldsymbol{z}_{L-\{\ell\}} \to 0 \\ \boldsymbol{z}_{L-\{\ell\}} \in T_{L-\{\ell\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{\ell\}}, \boldsymbol{0}_{L^c})} f_{\ell\alpha_\ell}(\boldsymbol{z}_{\ell^c})}{\boldsymbol{\alpha}_{L-\{\ell\}}!};$$

the limits are uniform on bounded proper subpolysectors of S_{L^c} whenever $L \neq N$.

$TA'(f)$ determines $TA(f)$ uniquely. Conversely, if we consider a family

$$\mathcal{F}' = \{ f_{jm} \in \mathcal{A}(S_{j^c}, E) \colon j \in N,\ m \in \mathbb{N} \}$$

under the first order consistency conditions (we will say $\mathcal{F}' = \{ f_{jm} \}$ is a *consistent first order family*), we may construct in a unique way a consistent family $\mathcal{F} = \{ f_{\boldsymbol{\alpha}_J} \}$ whose first order subfamily is $\mathcal{F}'$ (for details, see [9]).

Now the solution is given by the next result.

Theorem 4.4 *Given a consistent first order family $\mathcal{G} = \{f_{jm}\}$, there exists $f \in \mathcal{A}(S, E)$ such that $TA'(f) = \mathcal{G}$.*

Proof: We give the proof for $n = 3$, since this suffices to show how to proceed in the general case. So,

$$\mathcal{G} = \{f_{jm} \in \mathcal{A}(S_{j^c}, E) \colon j \in \{1, 2, 3\},\ m \in \mathbb{N}\} = \{f_{1m},\ f_{2m},\ f_{3m} \colon m \in \mathbb{N}\}.$$

Consider the sequence $\{f_{1m}\}_{m\in\mathbb{N}}$ in $\mathcal{A}(S_{1^c}, E)$; Proposition 4.1 assures there exists $H_1^{[1]*} \in \mathcal{A}(S_1, \mathcal{A}(S_{1^c}, E))$ with $H_1^{[1]*} \sim \sum_{m=0}^{\infty} f_{1m} z_1^m$.

By Theorem 3.3 the function $H^{[1]}$ given by $H^{[1]}(\boldsymbol{z}) = H_1^{[1]*}(z_1)(\boldsymbol{z}_{1^c})$ belongs to $\mathcal{A}(S, E)$; put $TA'(H^{[1]}) = \{h_{jm}^{[1]}\}$. For $T_1 \ll S_1$ and every $\boldsymbol{z}_{1^c} \in S_{1^c}$ we have

$$h_{1m}^{[1]}(\boldsymbol{z}_{1^c}) = \lim_{\substack{z_1\to 0\\ z_1\in T_1}} \frac{D^{me_1} H^{[1]}(\boldsymbol{z})}{m!} = \lim_{\substack{z_1\to 0\\ z_1\in T_1}} \frac{(H_1^{[1]*})^{m)}(z_1)(\boldsymbol{z}_{1^c})}{m!} = f_{1m}(\boldsymbol{z}_{1^c}).$$

Let us consider the function $H_2^{[1]*}$ given by

$$H_2^{[1]*}(z_2)(\boldsymbol{z}_{2^c}) = H^{[1]}(z_2, \boldsymbol{z}_{2^c}), \qquad z_2 \in S_2,\ \boldsymbol{z}_{2^c} \in S_{2^c}.$$

By Theorem 3.3, $H_2^{[1]*} \in \mathcal{A}(S_2, \mathcal{A}(S_{2^c}, E))$ and $H_2^{[1]*} \sim \sum_{m=0}^{\infty} h_{2m}^{[1]} z_2^m$. From the coherence conditions for $\mathcal{G}$ and $TA'(H^{[1]})$ we have that for every $m, k \in \mathbb{N}$, and $T_1 \ll S_1$, $T_2 \ll S_2$,

$$\lim_{\substack{z_1\to 0\\ z_1\in T_1}} \frac{D^{me_1}(f_{2k} - h_{2k}^{[1]})(\boldsymbol{z}_{2^c})}{m!} = \lim_{\substack{z_2\to 0\\ z_2\in T_2}} \frac{D^{ke_2}(f_{1m} - h_{1m}^{[1]})(\boldsymbol{z}_{1^c})}{k!} = 0.$$

We can apply Lemma 4.3 to obtain a function $H_2^{[2]*} \in \mathcal{A}(S_2, \mathcal{A}(S_{2^c}, E))$ with

$$H_2^{[2]*} \sim \sum_{m=0}^{\infty} (f_{2m} - h_{2m}^{[1]}) z_2^m. \tag{4.3}$$

Theorem 3.3 implies that $H^{[2]}$, given by $H^{[2]}(\boldsymbol{z}) = H_2^{[2]*}(z_2)(\boldsymbol{z}_{2^c})$, belongs to $\mathcal{A}(S,E)$; if we put $TA'(H^{[2]}) = \{h_{jm}^{[2]}\}$, by Lemma 4.3 and (4.3) we have

$$h_{1m}^{[2]} = 0, \quad h_{2m}^{[2]} = f_{2m} - h_{2m}^{[1]}, \quad m \in \mathbb{N}.$$

So, $F^{[2]} = H^{[1]} + H^{[2]}$ is in $\mathcal{A}(S,E)$ and, if $TA'(F^{[2]}) = \{f_{jm}^{[2]}\}$, we have $f_{jm}^{[2]} = f_{jm}$ for $j = 1,2$ and $m \in \mathbb{N}$. Consider the function $F_3^{[2]*}$ defined by

$$F_3^{[2]*}(z_3)(\boldsymbol{z}_{3^c}) = F^{[2]}(z_3, \boldsymbol{z}_{3^c}), \qquad z_3 \in S_3,\ \boldsymbol{z}_{3^c} \in S_{3^c}.$$

According to Theorem 3.3, $F_3^{[2]*} \in \mathcal{A}(S_3, \mathcal{A}(S_{3^c}, E))$ and $F_3^{[2]*} \sim \sum_{m=0}^{\infty} f_{3m}^{[2]} z_3^m$. From the consistency conditions for $\mathcal{G}$ and $TA'(F^{[2]})$ we have that for $T = \prod_{j=1}^{3} T_j \ll S$, for every $m, k \in \mathbb{N}$ and for $j = 1,2$,

$$\lim_{\substack{z_j \to 0 \\ z_j \in T_j}} \frac{D^{me_j}(f_{3k} - f_{3k}^{[2]})(\boldsymbol{z}_{3^c})}{m!} = \lim_{\substack{z_3 \to 0 \\ z_3 \in T_3}} \frac{D^{ke_3}(f_{jm} - f_{jm}^{[2]})(\boldsymbol{z}_{j^c})}{k!} = 0.$$

We can apply Lemma 4.3 to obtain $H_3^{[3]*} \in \mathcal{A}(S_3, \mathcal{A}(S_{3^c}, E))$ such that

$$H_3^{[3]*} \sim \sum_{m=0}^{\infty} (f_{3m} - f_{3m}^{[2]}) z_3^m. \tag{4.4}$$

Again by Theorem 3.3, the function $H^{[3]}$ given by $H^{[3]}(\boldsymbol{z}) = H_3^{[3]*}(z_3)(\boldsymbol{z}_{3^c})$ belongs to $\mathcal{A}(S,E)$, and if $TA'(H^{[3]}) = \{h_{jm}^{[3]}\}$, then Lemma 4.3 and (4.4) imply

$$\begin{aligned} h_{jm}^{[3]} &= 0, \quad j = 1,2,\ m \in \mathbb{N}; \\ h_{3m}^{[3]} &= f_{3m} - f_{3m}^{[2]}, \quad m \in \mathbb{N}. \end{aligned}$$

So, $F^{[3]} = F^{[2]} + H^{[3]}$ is our solution. □

Finally, we present a new proof of the classical theorem of E. Borel, for vector-valued functions, based on the asymptotic techniques provided by the previous results.

Theorem 4.5 *Let $\{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ be a multisequence in E. Then, there exists a $\mathcal{C}^\infty$ function g from $\mathbb{R}^n$ to E such that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\dfrac{D^{\boldsymbol{\alpha}} g(\mathbf{0})}{\boldsymbol{\alpha}!} = a_{\boldsymbol{\alpha}}$.*

Proof: Consider the polysectors $S = \prod_{j=1}^{n} S_j$ and $T = \prod_{j=1}^{n} T_j$ in $\mathbb{C}^n$, where, for $j = 1, 2, \ldots, n$, we put

$$S_j = \{\, z \in \mathbb{C} \colon -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2} \,\},$$

$$T_j = \{\, z \in \mathbb{C}\colon -\frac{\pi}{4} < \arg(z) < \frac{5\pi}{4} \,\}.$$

Observe that $T \ll S$ and $\{\, z \in \mathbb{C}\colon \mathrm{Im}(z) \geq 0,\ z \neq 0 \,\} \subset T_j$. We can apply Theorem 4.2 to obtain a function $f \in \mathcal{A}(S,E)$ such that $f \sim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$. Then, for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ we have

$$\lim_{\substack{\boldsymbol{z}\to 0\\ \boldsymbol{z}\in T}} \frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{z})}{\boldsymbol{\alpha}!} = a_{\boldsymbol{\alpha}}.$$

For $m \in \mathbb{N}$, $m \geq 1$, let us define the function $f_m\colon \mathbb{R}^n \to E$ by

$$f_m(\boldsymbol{x}) = f_m(x_1,\ldots,x_n) = f(x_1 + \frac{i}{m},\ldots,x_n + \frac{i}{m}), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Obviously, f_m is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$. We will prove that for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ the sequence $\{D^{\boldsymbol{\alpha}} f_m\}_{m=1}^\infty = \{\dfrac{\partial^{\boldsymbol{\alpha}} f_m}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}\}_{m=1}^\infty$ converges uniformly on the compact subsets of $\mathbb{R}^n$. Indeed, consider the compact set $I_r = [-r,r]^n$, $r > 0$. For every $m \geq 1$, the set

$$\{\, (x_1 + \frac{i}{m},\ldots,x_n + \frac{i}{m})\colon (x_1,\ldots,x_n) \in I_r \,\}$$

is contained in T. By the chain rule and since f is holomorphic in T, for $\boldsymbol{x} = (x_1,\ldots,x_n) \in I_r$ we have

$$\frac{\partial^{\boldsymbol{\alpha}} f_m}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) = \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(x_1 + \frac{i}{m},\ldots,x_n + \frac{i}{m}) = \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}}(x_1 + \frac{i}{m},\ldots,x_n + \frac{i}{m}), \tag{4.5}$$

and hence, for any natural numbers $m, \ell \geq 1$ and any cs p on E we may write

$$\begin{aligned} &p(\frac{\partial^{\boldsymbol{\alpha}} f_m}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) - \frac{\partial^{\boldsymbol{\alpha}} f_\ell}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}))\\ &= p\Big(\sum_{j=1}^n \int_{x_j+\frac{1}{\ell}}^{x_j+\frac{1}{m}} D^{\boldsymbol{\alpha}+e_j} f(x_1 + \frac{i}{m},\ldots,x_{j-1} + \frac{i}{m}, t, x_{j+1} + \frac{i}{\ell},\ldots,x_n + \frac{i}{\ell})\,dt\Big)\\ &\leq |\frac{1}{\ell} - \frac{1}{m}| \sum_{j=1}^n \sup_{\boldsymbol{z}\in T,\ |z_j|<r+2} p(D^{\boldsymbol{\alpha}+e_j} f(\boldsymbol{z})), \qquad \boldsymbol{x} \in I_r, \end{aligned}$$

from where the uniform convergence on I_r is deduced.

A classical theorem for the convergence of a sequence of $\mathcal{C}^\infty$ functions allows to conclude that, if we define the function $g\colon \mathbb{R}^n \to E$ as

$$g(\boldsymbol{x}) = \lim_{m\to\infty} f_m(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

then g is a $\mathcal{C}^\infty$ function, and for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ and every $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$D^{\boldsymbol{\alpha}} g(\boldsymbol{x}) = \lim_{m\to\infty} D^{\boldsymbol{\alpha}} f_m(\boldsymbol{x}).$$

Taking $\boldsymbol{x} = \mathbf{0}$ and applying (4.5), we get

$$D^{\boldsymbol{\alpha}} g(\mathbf{0}) = \lim_{m\to\infty} D^{\boldsymbol{\alpha}} f_m(\mathbf{0}) = \lim_{m\to\infty} D^{\boldsymbol{\alpha}} f(\frac{i}{m},\dots,\frac{i}{m}) = \lim_{\substack{\boldsymbol{z}\to 0\\ \boldsymbol{z}\in T}} D^{\boldsymbol{\alpha}} f(\boldsymbol{z}) = \boldsymbol{\alpha}!\, a_{\boldsymbol{\alpha}},$$

as desired. □

5 Other concepts of asymptotic expansion

Gérard and Jurkat [1] introduce the space $\mathcal{A}(\varphi)$ of (germs of) complex (not necessarily holomorphic) functions with asymptotic expansion with respect to an asymptotic sequence φ. We refer the reader to their paper for the details, and note that the freedom of choice of some of the elements entering the definition of $\mathcal{A}(\varphi)$ makes it available for a wide class of situations. The preparation theorem and the division theorem are obtained for the space $\mathcal{A}(\varphi)\{y\}$ of complex functions g of the form $\sum_{m=0}^{\infty} g_m(x)y^m$, where the $g_m \in \mathcal{A}(\varphi)$, are defined in a common domain D_g, and are subject to bounds that assure normal convergence of the series in $D_g \times \{y \in \mathbb{C}\colon |y| \le r_g\}$, for some suitable $r_g > 0$. The asymmetric roles played by the variables x and y are due to the fact that y is the "division" variable, while x is the "asymptotic" one.

In order to translate the previous setting into the context of this paper, we restrict our attention to holomorphic functions, and treat only the two variables case, which is both easier to handle and enlightening enough. For a sector S, the space of complex holomorphic functions $f\colon S \to \mathbb{C}$ admitting a power series asymptotic expansion is just $\mathcal{A}(S,\mathbb{C})$ (for short, $\mathcal{A}(S)$), as defined in Section 3; the same is valid when changing $\mathbb{C}$ for a Fréchet space E. So, for $D = \{\omega \in \mathbb{C}\colon |\omega| < r\}$, and as an equivalent to $\mathcal{A}(\varphi)\{y\}$, one may consider the space $\mathcal{M}_1$ of holomorphic functions $f\colon S \times D \to \mathbb{C}$ such that, when we write

$$f(z,\omega) = \sum_{m=0}^{\infty} g_m(z)\omega^m, \qquad (z,\omega) \in S \times D,$$

it holds that $g_m \in \mathcal{A}(S)$ for every $m \in \mathbb{N}$.

Functions in $\mathcal{M}_1$ can be quite awkward; in particular, for fixed $\omega \in D$, $f(\cdot,\omega)$ need not admit an asymptotic expansion in z (see Problem 9.4 in [11, p. 47]). Different (and in fact, more demanding) approaches may be taken, for example:

(i) Following Wasow [11, Ch. III, Sc. 9.3], define the space $\mathcal{M}_2$ of holomorphic functions $f\colon S \times D \to \mathbb{C}$ admitting an asymptotic expansion in z, $f(z,\omega) \sim_S \sum_{n=0}^{\infty} f_n(\omega)z^n$, uniformly on every compact $K \subset D$. This means that for every such K, every $T \ll S$ and every $n \in \mathbb{N}$,

$$\sup_{\omega \in K} \sup_{z \in T} \left| \frac{f(z,\omega) - \sum_{\ell=0}^{n-1} f_\ell(\omega) z^\ell}{z^n} \right| < \infty.$$

(ii) Since $\mathcal{H}(D)$, space of complex holomorphic functions in D, is a Fréchet space when given its compact-open topology, one can consider the space $\mathcal{A}(S, \mathcal{H}(D))$, which may be regarded as a subspace of $\mathcal{H}(S \times D)$.

The very definition of $\mathcal{A}(S, \mathcal{H}(D))$, together with Theorems 9.4 and 9.5 and Problem 9.4 in [11], let us conclude that $\mathcal{A}(S, \mathcal{H}(D)) = \mathcal{M}_2 \subsetneq \mathcal{M}_1$. It is worth noting that these spaces can be characterized in terms of bounds for the derivatives of their elements:

$$\begin{aligned} f \in \mathcal{M}_1 \quad &\iff \quad \text{for every } T \ll S, \quad n,\, m \in \mathbb{N}, \quad \sup_{z \in T} |D^{(n,m)} f(z,0)| < \infty; \\ f \in \mathcal{M}_2 \quad &\iff \quad \text{for every compact } K \subset D, \quad T \ll S, \quad n \in \mathbb{N}, \\ &\qquad\quad \sup_{(z,\omega) \in T \times K} |D^{(n,0)} f(z,\omega)| < \infty \\ &\iff \quad \text{for every compact } K \subset D, \quad T \ll S, \quad n,\, m \in \mathbb{N}, \\ &\qquad\quad \sup_{(z,\omega) \in T \times K} |D^{(n,m)} f(z,\omega)| < \infty. \end{aligned}$$

When changing the disk D for a second sector U, there are many possible choices for the space to be considered, some of which are the analogue ones to the previously studied. Since the variables z and ω are now interchangeable, we do not consider symmetrical situations. The following are all defined as subspaces of $\mathcal{H}(S \times U)$:

(i) $f \in \mathcal{N}_1$ if for every $z \in S$ one has an expansion $f(z,\omega) \sim_U \sum_{m=0}^{\infty} g_m(z) \omega^m$, and $g_m \in \mathcal{A}(S)$ for every $m \in \mathbb{N}$.

(ii) $f \in \mathcal{N}_2$ if $f(z,\omega) \sim_U \sum_{m=0}^{\infty} g_m(z) \omega^m$, uniformly on every $T \ll S$, and $g_m \in \mathcal{A}(S)$ for every $m \in \mathbb{N}$.

(iii) $f \in \mathcal{N}_{GS}$ if it admits Gérard-Sibuya asymptotic expansion in $S \times U$.

(iv) $f \in \mathcal{N}_3$ if there exist $h_n \in \mathcal{A}(U)$, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ the function $\dfrac{1}{z^n}[f(z,\omega) - \sum_{\ell=0}^{n-1} h_\ell(\omega) z^\ell]$ admits asymptotic expansion in U, uniformly on every $T \ll S$.

(v) The space $\mathcal{A}(S, \mathcal{A}(U))$, that, according to the results in Section 3, coincides (up to isomorphism) with $\mathcal{A}(S \times U)$, with the space of strongly asymptotically developable functions in $S \times U$, and with the space of functions whose derivatives all admit Gérard-Sibuya asymptotic expansion.

The introduction of (iv) comes from Sibuya [10], who used a similar kind of expansions; Gérard and Sibuya also commented on such a case [2, p. 172], and Majima [5, p. 7] noticed that these expansions almost coincide with his concept of strongly asymptotic expansion. With the information in Section 3 (in particular, on the second family of seminorms defining the topology of $\mathcal{A}(S, E)$), one can obtain that $\mathcal{A}(S \times U) = \mathcal{A}(S, \mathcal{A}(U)) = \mathcal{N}_3$, and that for $f \in \mathcal{N}_3$ the functions $h_n \in \mathcal{A}(U)$ are just the elements of $TA(f)$ that allow to write $f \equiv f^* \sim_S h_n z^n$.

Of course, $\mathcal{N}_3 \subset \mathcal{N}_2 \subset \mathcal{N}_1$; as indicated in [2, p. 171], we also have $\mathcal{N}_2 \subset \mathcal{N}_{GS}$. Examples can be given to show that $\mathcal{N}_2 \not\subset \mathcal{N}_3$, $\mathcal{N}_{GS} \not\subset \mathcal{N}_2$, $\mathcal{N}_1 \not\subset \mathcal{N}_2$, $\mathcal{N}_{GS} \not\subset \mathcal{N}_1$ and $\mathcal{N}_1 \not\subset \mathcal{N}_{GS}$.

Finally, we mention that the different spaces considered may be characterized as well in terms of bounds for the derivatives, in the same way as $\mathcal{N}_3$. For example, $f \in \mathcal{N}_2$ if and only if: (i) for every $T \ll S$, $V \ll U$ and $m \in \mathbb{N}$,

$$\sup_{(z,\omega) \in T \times V} |D^{(0,m)} f(z, \omega)| < \infty;$$

(ii) the functions g_m, given by

$$g_m(z) = \lim_{\omega \to 0,\ \omega \in V} \frac{D^{(0,m)} f(z, \omega)}{m!}, \qquad m \in \mathbb{N},\ z \in S,$$

are such that for every $T \ll S$ and $n \in \mathbb{N}$, $\sup_{z \in T} |g_m^{n)}(z)| < \infty$.

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