# Extension operators
# in Carleman ultraholomorphic classes

by

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**Abstract.** Linear and continuous extension operators are constructed in Carleman ultraholomorphic classes of functions of several variables in polysectors, so generalizing Borel-Ritt-Gevrey theorem. With the help of quasi-analyticity results, some rigidity properties of these operators are analyzed.

## 1 Introduction

Given $A > 0$, a sequence of positive real numbers $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ and a sector $S$ with vertex at 0 in the Riemann surface of the logarithm, $\mathcal{R}$, we define $\mathcal{A}_{\mathbf{M},A}(S)$ as the set of complex holomorphic functions $f$ defined in $S$ such that

$$\|f\|_{\mathbf{M},A,S} := \sup_{p\in\mathbb{N}_0, z\in S} \frac{|D^p f(z)|}{A^p p! M_p} < \infty.$$

Accordingly, $\Lambda_{\mathbf{M},A}(\mathbb{N}_0)$ consists of the sequences of complex numbers $\boldsymbol{\lambda} = (\lambda_p)_{p\in\mathbb{N}_0}$ such that

$$|\boldsymbol{\lambda}|_{\mathbf{M},A} := \sup_{p\in\mathbb{N}_0} \frac{|\lambda_p|}{A^p p! M_p} < \infty.$$

$(\mathcal{A}_{\mathbf{M},A}(S), \|\cdot\|_{\mathbf{M},A,S})$ and $(\Lambda_{\mathbf{M},A}(\mathbb{N}_0), |\cdot|_{\mathbf{M},A})$ are Banach spaces, and we may consider the so-called Borel map $\mathcal{B} : \mathcal{A}_{\mathbf{M},A}(S) \to \Lambda_{\mathbf{M},A}(\mathbb{N}_0)$ given by

$$\mathcal{B}(f) := \big(f^{(p)}(0) := \lim_{z\to 0} f^{(p)}(z)\big)_{p\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}.$$

$\mathcal{B}$ is well defined, linear and continuous, and it is still so when defined between the inductive limits $\mathcal{A}_{\mathbf{M}}(S) := \cup_{A>0}\mathcal{A}_{\mathbf{M},A}(S)$ and $\Lambda_{\mathbf{M}}(\mathbb{N}_0) := \cup_{A>0}\Lambda_{\mathbf{M},A}(\mathbb{N}_0)$, the first of which is usually named an ultraholomorphic Carleman class. It is worth noticing that the functions $f \in \mathcal{A}_{\mathbf{M}}(S)$ admit asymptotic expansion at 0, namely $f \sim \sum_{p\in\mathbb{N}_0} \frac{1}{p!} f^{(p)}(0) z^p$.

At this point, the question arises whether it is possible to construct extension operators, that is, linear and continuous right inverses for $\mathcal{B}$ in suitable subspaces of $\Lambda_{\mathbf{M}}(\mathbb{N}_0)$ and $\mathcal{A}_{\mathbf{M}}(S)$, respectively endowed with natural topologies. The first answer was given for Gevrey classes of order $\alpha > 1$ in a sector $S$, corresponding to the sequence $\mathbf{M}_\alpha = (p!^{\alpha-1})_{p\in\mathbb{N}_0}$, which constantly appear in the theory of algebraic ordinary differential equations, of meromorphic, linear or not, systems of ordinary differential equations at an irregular singular point, and of partial differential equations. The by-now classical Borel-Ritt-Gevrey theorem, obtained by J.-P. Ramis [14] and stating the surjectivity of $\mathcal{B} : \mathcal{A}_{\mathbf{M}_\alpha}(S) \to \Lambda_{\mathbf{M}_\alpha}(\mathbb{N}_0)$ whenever the opening of the sector $S$ is at most $(\alpha-1)\pi$, was generalized through the construction of extension operators by V. Thilliez [17] and, some time later and by means of a detailed study of the integral solution of Ramis, by the second author [15]. Recently, a more general result has been given by V. Thilliez [18] for strongly regular sequences $\mathbf{M}$ (see Subsection 2.2), what includes the Gevrey case. Although he works with complex-valued functions, there is no difficulty in adapting the result for functions with values in a complex Banach space $B$ (with the natural modifications in the definition of the spaces of $B$-valued functions, $\mathcal{A}_{\mathbf{M},A}(S,B)$, and of sequences of elements in $B$, $\Lambda_{\mathbf{M},A}(\mathbb{N}_0,B)$)).

**Theorem 1.1** ([18], Theorem 3.2.1)**.** Let $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence with associated growth index $\gamma(\mathbf{M})$. Let us consider $\gamma \in \mathbb{R}$ with $0 < \gamma < \gamma(\mathbf{M})$, and let $S_\gamma$ be a sector with opening $\gamma$. Then there exists $d \geq 1$, that only depends on $\mathbf{M}$ and $\gamma$, so that for every $A > 0$ there exists a linear continuous operator

$$T_{\mathbf{M},A,\gamma} : \Lambda_{\mathbf{M},A}(\mathbb{N}_0,B) \longrightarrow \mathcal{A}_{\mathbf{M},dA}(S_\gamma,B)$$

such that $\mathcal{B} \circ T_{\mathbf{M},A,\gamma}\boldsymbol{\lambda} = \boldsymbol{\lambda}$ for every $\boldsymbol{\lambda} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0,B)$.

Our main aim in this paper is to obtain similar extension operators in the case of several variables. For a polysector (cartesian product of sectors) $S$ in $\mathcal{R}^n$, the space $\mathcal{A}_{\mathbf{M}}(S,B)$ consists of the holomorphic functions $f : S \to (B, \|\cdot\|_B)$ such that there exists $A > 0$ (depending on $f$) with

$$\|f\|^B_{\mathbf{M},A,S} := \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^n, \boldsymbol{z}\in S} \frac{\|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\|_B}{A^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|}} < \infty. \tag{1}$$

For $A > 0$, $\mathcal{A}_{\mathbf{M},A}(S,B)$ is defined as before and the norm in (1) makes it a Banach space. The sets of multi-sequences $\Lambda_{\mathbf{M}}(\mathbb{N}_0^n, B)$ and $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ are similarly defined and topologized (see Definition 2.6). It turns out (see Subsection 2.4) that an element $f$ in $\mathcal{A}_{\mathbf{M}}(S,B)$ admits strong asymptotic development in $S$, as defined by H. Majima in 1983 [10, 11], and one may associate to it a unique family $\mathrm{TA}(f)$ consisting of functions obtained, as in the one variable case, as limits of the derivatives of $f$ when some of its variables tend to zero (see (5)). The elements of $\mathrm{TA}(f)$ admit strong asymptotic expansion in the corresponding polysector, are linked by certain coherence conditions (see (6)) and satisfy estimates deduced from those for $f$. We write $\mathfrak{F}_{\mathbf{M}}(S,B)$ for the set of all such coherent families in $S$ and consider the maps

$$\mathcal{B} : \mathcal{A}_{\mathbf{M}}(S,B) \to \Lambda_{\mathbf{M}}(\mathbb{N}_0^n, B) \qquad \text{and} \qquad \mathrm{TA} : \mathcal{A}_{\mathbf{M}}(S,B) \to \mathfrak{F}_{\mathbf{M}}(S,B),$$

where the first one is defined as $\mathcal{B}(f) := \big(D^{\boldsymbol{\alpha}} f(\mathbf{0}) := \lim_{\boldsymbol{z}\to\mathbf{0}} D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\big)_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}$. We will prove the existence of right inverses for both maps.

Some results in this direction have appeared in the literature, though they deal with slightly different classes. In 1989 Y. Haraoka [4] considered the space $\mathcal{A}_{\boldsymbol{\alpha},A}(S)$ of holomorphic functions $f$ in $S$ that admit Gevrey strong asymptotic expansion of order $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in [1,\infty)^n$ (one order per variable) in a bounded polysector $S$ in $\mathbb{C}^n$ and got some Borel-Ritt-Gevrey type results. The second author [15] built linear and continuous extension operators in the same context, heavily resting on the fact that the one-dimensional solution in integral form can be easily adapted in order to obtain functions with values on a Banach space with preassigned asymptotic behaviour in a sector. This allowed him to reduce the number of variables via an isomorphism of the type

$$\mathcal{A}_{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2),(A_1,A_2)}(S_1 \times S_2) \simeq \mathcal{A}_{\boldsymbol{\alpha}_1,A_1}(S_1, \mathcal{A}_{\boldsymbol{\alpha}_2,A_2}(S_2)). \tag{2}$$

This idea will also be useful now, and that is why we consider Banach space-valued functions. In Section 3 we establish linear and continuous maps (not isomorphisms as in (2), but that makes no difference) that allow us to go from one side to the other in a correspondence similar to (2) (see Theorem 3.1). This is the key for reducing the number of variables in our problem to only one. By an inductive reasoning, we deduce the existence of extension operators for $\mathcal{B}$ (Theorem 3.2) as long as the opening $\pi\gamma_j$ of every factor $S_j$ in $S$ is such that $\gamma_j < \gamma(\mathbf{M})$. In order to tackle the corresponding problem for TA, we will reformulate it in terms of the so-called first order family $\mathcal{B}_1(f)$ associated to each element $f$ in $\mathcal{A}_{\mathbf{M},A}(S,B)$, that, being a subfamily of $\mathrm{TA}(f)$, completely determines it thanks to the coherence conditions. After finding an adequate space of first order families where interpolation is possible (Proposition 3.4), we get in Theorem 3.6 the second extension operator by means of a recursive process, in which it turns out to be fundamental the study, carried out in Lemma 3.5, of the behaviour of the derivatives of the solution of the one dimensional problem when it takes its values in a Banach space of the type $\mathcal{A}_{\mathbf{M},A}(S,B)$. Unlike the Gevrey case, this study is now hard, since the construction of Thilliez's operators is based on a double application of the Whitney's extension operators given by J. Chaumat and A.-M. Chollet [2] (see Subsection 2.3) and on a solution of a $\overline{\partial}$-problem. Due to its technical character and length, the proof of Lemma 3.5 will be only sketched and postponed till Section 5.

To end this Section, we also give some results about the necessity of the condition $\max_{1\le j\le n}\gamma_j < \gamma(\mathbf{M})$ for the existence of the extension operators we have constructed.

In Section 4 we show some results (Theorems 4.2 and 4.4) on the rigidity of the extension operators obtained, consisting of the determination of annihilation conditions on the interpolating functions which ensure the initial data were null. We adopt the same line of argument as in the results given by V. Thilliez [17] for the one variable case. We also indicate how to combine these results with quasi-analyticity properties. We say $\mathcal{A}_{\mathbf{M}}(S)$ is quasi-analytic (respectively, (s) quasi-analytic) whenever $\mathcal{B}$ (resp. TA) is injective. Both concepts have been characterized by the authors [9] in terms of Watson's type lemmas (as indicated in Subsection 2.5) which can be applied if the opening of the polysectors is adequate (see Corollaries 4.3 and 4.5).

The extension operators considered in this paper have been already applied by the authors, in the one dimensional case, to the Stieltjes moment problem for general Gelfand-Shilov spaces [7, 8]. We expect that the multidimensional results given here also find their application in this context.

# 2 Preliminaries

## 2.1 Notations

$\mathbb{N}$ will stand for $\{1, 2, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we put $\mathcal{N} = \{1, 2, \ldots, n\}$. If $J$ is a nonempty subset of $\mathcal{N}$, $\#J$ denotes its cardinal number.

We will consider sectors in the Riemann surface of the logarithm $\mathcal{R}$ with vertex at 0. Let $\theta > 0$. We will write $S_\theta = \left\{z : |\arg z| < \frac{\theta\pi}{2}\right\}$, the sector of opening $\theta\pi$ and bisecting direction $d = 0$.

Let $S$ be a sector. A proper subsector $T$ of $S$ is a sector such that $\overline{T} \setminus \{0\} \subseteq S$. If moreover $T$ is bounded, we say $T$ is a bounded proper subsector of $S$, and write $T \prec S$.

A polysector is a cartesian product $S = \prod_{j=1}^n S_j \subset \mathcal{R}^n$ of sectors. A polysector $T$ is a proper subpolysector of $S$ if $T = \prod_{j=1}^n T_j$ with $\overline{T}_j \setminus \{0\} \subseteq S_j$, $j = 1, 2, \ldots, n$. $T$ is bounded if each one of its factors is.

Given $\boldsymbol{z} \in \mathcal{R}^n$, we write $\boldsymbol{z}_J$ for the restriction of $\boldsymbol{z}$ to $J$, regarding $\boldsymbol{z}$ as an element of $\mathcal{R}^{\mathcal{N}}$.

Let $J$ and $L$ be nonempty disjoint subsets of $\mathcal{N}$. For $\boldsymbol{z}_J \in \mathcal{R}^J$ and $\boldsymbol{z}_L \in \mathcal{R}^L$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)$ represents the element of $\mathcal{R}^{J\cup L}$ satisfying $(\boldsymbol{z}_J, \boldsymbol{z}_L)_J = \boldsymbol{z}_J$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)_L = \boldsymbol{z}_L$; we also write $J' = \mathcal{N} \setminus J$, and for $j \in \mathcal{N}$ we use $j'$ instead of $\{j\}'$. In particular, we shall use these conventions for multi-indices.

For $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in (0, \infty)^n$, we write $S_{\boldsymbol{\theta}} = \prod_{j=1}^n S_{\theta_j}$ and $S_{\boldsymbol{\theta}_J} = \prod_{j\in J} S_{\theta_j} \subset \mathcal{R}^J$.

If $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$, we define:

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \qquad \boldsymbol{\alpha}! = \alpha_1!\alpha_2! \cdots \alpha_n!,$$
$$\boldsymbol{D}^{\boldsymbol{\alpha}} = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \ldots \partial z_n^{\alpha_n}}, \quad \boldsymbol{e}_j = (0, \ldots, \overset{j)}{1}, \ldots, 0).$$

For $\boldsymbol{J} \in \mathbb{N}_0^n$, we will frequently write $j = |\boldsymbol{J}|$.

$f \sim \sum_{p\in\mathbb{N}_0} a_p z^p$ means that the series $\sum_{p\in\mathbb{N}_0} a_p z^p$ is the asymptotic expansion of the function $f$ at 0.

## 2.2 Strongly regular sequences

In what follows, $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ will always stand for a sequence of positive real numbers, and we will always assume that $M_0 = 1$.

**Definition 2.1.** We say $\mathbf{M}$ is *strongly regular* if the following hold:
($\alpha_0$) $\mathbf{M}$ is *logarithmically convex*: $M_n^2 \leq M_{n-1}M_{n+1}$ for every $n \in \mathbb{N}$.
($\mu$) $\mathbf{M}$ is *of moderate growth*: there exists $A > 0$ such that

$$M_{p+\ell} \leq A^{p+\ell} M_p M_\ell, \qquad p, \ell \in \mathbb{N}_0.$$

($\gamma_1$) $\mathbf{M}$ satisfies the *strong non-quasianalyticity condition*: there exists $B > 0$ such that

$$\sum_{\ell\geq p} \frac{M_\ell}{(\ell+1)M_{\ell+1}} \leq B \frac{M_p}{M_{p+1}}, \qquad p \in \mathbb{N}_0.$$

The measurable function $T_{\mathbf{M}} : (0, \infty) \to (0, \infty]$ is given by

$$T_{\mathbf{M}}(r) = \sup_{p\in\mathbb{N}_0} \frac{r^p}{M_p}, \qquad r > 0. \tag{3}$$

Following V. Thilliez [18], we define next the growth index of a strongly regular sequence.

**Definition 2.2.** Let $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence, $\gamma > 0$. We say $\mathbf{M}$ satisfies property $(P_\gamma)$ if there exist a sequence of real numbers $m' = (m'_p)_{p\in\mathbb{N}}$ and a constant $a \geq 1$ such that: (i) $a^{-1}M_p \leq M_{p-1}m'_p \leq aM_p$, $p \in \mathbb{N}$, and (ii) $\big((p+1)^{-\gamma}m'_p\big)_{p\in\mathbb{N}}$ is increasing.

**Definition 2.3.** Let $\mathbf{M}$ be a strongly regular sequence. The *growth index* of $\mathbf{M}$ is

$$\gamma(\mathbf{M}) = \sup\{\gamma \in \mathbb{R} : (P_\gamma) \text{ is fulfilled}\}.$$

**Remark 2.4.** It may be proved that $\gamma(\mathbf{M}) \in (0,\infty)$. For the Gevrey sequence of order $\alpha > 0$, $\mathbf{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, we have $\gamma(\mathbf{M}_\alpha) = \alpha$.

## 2.3 Whitney type results in ultradifferentiable classes

We comment here on the Whitney type result obtained by J. Chaumat and A.-M. Chollet [2] which will be useful for our purposes. Although it was originally given for functions and jets with values in $\mathbb{C}$, its generalization to function and jet spaces with values in a general Banach space does not offer any difficulty.

From now on, $\Omega$ will be a nonempty open set in $\mathbb{R}^n$ and $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ a sequence of positive real numbers.

**Definition 2.5.** We say a map $f \in \mathcal{C}^\infty(\mathbb{R}^n, B)$ belongs to the *Carleman class* $\mathcal{C}_{\mathbf{M}}(\Omega, B)$ if there exists a constant $A = A(f) > 0$ such that

$$\sup_{\boldsymbol{J}\in\mathbb{N}_0^n,\boldsymbol{x}\in\Omega} \frac{\left\|D^{\boldsymbol{J}} f(\boldsymbol{x})\right\|_B}{A^j j! M_j} < \infty. \tag{4}$$

The set consisting of the funtions in $\mathcal{C}_{\mathbf{M}}(\Omega, B)$ that fulfill (4) for a fixed $A > 0$ is denoted $\mathcal{C}_{\mathbf{M},A}(\Omega, B)$, and it is a Banach space when endowed with the norm in (4), say $\|\cdot\|^B_{\mathbf{M},A,\Omega}$ (this agrees with (1)).

For $B = \mathbb{C}$, we write $\mathcal{C}_{\mathbf{M},A}(\Omega,\mathbb{C}) =: \mathcal{C}_{\mathbf{M},A}(\Omega)$, $\mathcal{C}_{\mathbf{M}}(\Omega,\mathbb{C}) =: \mathcal{C}_{\mathbf{M}}(\Omega)$.

**Definition 2.6.** Given $A > 0$, the space $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ consists of the multi-sequences $\boldsymbol{\lambda} = (\lambda_{\boldsymbol{J}})_{\boldsymbol{J}\in\mathbb{N}_0^n}$ in $B$ such that

$$|\boldsymbol{\lambda}|_{\mathbf{M},A,B} := \sup_{\boldsymbol{J}\in\mathbb{N}_0^n} \frac{\|\lambda_{\boldsymbol{J}}\|_B}{A^j j! M_j} < \infty.$$

$(\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B), |\cdot|_{\mathbf{M},A,B})$ is a Banach space.

Again, we will write $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n) := \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,\mathbb{C})$, $\Lambda_{\mathbf{M}}(\mathbb{N}_0^n) := \Lambda_{\mathbf{M}}(\mathbb{N}_0^n,\mathbb{C})$.

It is easy to verify that, for $\boldsymbol{x}_0 \in \Omega$ fixed, the *Borel map*

$$\begin{aligned} \mathcal{B} : \mathcal{C}_{\mathbf{M},A}(\Omega, B) &\longrightarrow \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B) \\ f &\longrightarrow \mathcal{B}f := (D^{\boldsymbol{J}} f(\boldsymbol{x}_0))_{\boldsymbol{J}\in\mathbb{N}_0^n} \end{aligned}$$

is well defined, linear and continuous. Many authors have studied the possibility of constructing linear continuous right inverses for $\mathcal{B}$, and for the restriction map, to a compact set, of functions in $\mathcal{C}_{\mathbf{M}}(\mathbb{R}^n, B)$, so generalizing the classical Borel's and Whitney's theorems, respectively. H.-J. Petzsche [13] obtained definitive results in the first case, while J. Bonet, R. W. Braun, R. Meise and B. A. Taylor made prominent contributions in the second one (see [1] and the references therein). Our construction will rest upon the following result of J. Chaumat and A.-M. Chollet [2], that provides right inverses in both situations.

**Proposition 2.7.** Let $\mathbf{M}$ be a strongly regular sequence.

(i) There exists a constant $b \geq 1$, that only depends on $\mathbf{M}$ and $n$, such that for every $A > 0$, there exists a linear continuous operator

$$E_{A,B} : \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B) \longrightarrow \mathcal{C}_{\mathbf{M},bA}(\mathbb{R}^n, B)$$

such that $\mathcal{B}E_{A,B}\boldsymbol{\lambda} = \boldsymbol{\lambda}$ for every $\boldsymbol{\lambda}$ in $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$. Moreover, the extensions $E_{A,B}\boldsymbol{\lambda}$ can be built with compact support, contained in a fixed neighborhood of $\mathbf{0}$ that does not depend on $\boldsymbol{\lambda}$.

(ii) For every bounded and open set $\Omega$ in $\mathbb{R}^n$ with Lipschitz boundary, there exists a constant $c \geq 1$, depending on $\mathbf{M}$, $n$ and $\Omega$, such that for every $A > 0$ there exists a linear continuous operator

$$F_{A,\Omega,B} : \mathcal{C}_{\mathbf{M},A}(\Omega, B) \longrightarrow \mathcal{C}_{\mathbf{M},cA}(\mathbb{R}^n, B)$$

satisfying $F_{A,\Omega,B} f|_\Omega = f$ for every $f \in \mathcal{C}_{\mathbf{M},A}(\Omega, B)$. Moreover, the extension can be built with support contained in a fixed compact set in $\mathbb{R}^n$ (independent of $f$) that contains an open neighbourhood of the closure of $\Omega$.

## 2.4 Strong asymptotic expansions and ultraholomorphic classes in polysectors

Let $n \in \mathbb{N}$, $n \geq 2$, and $S$ a polysector in $\mathcal{R}^n$ with vertex at $\mathbf{0}$. H. Majima put forward the concept of strong asymptotic development for complex holomorphic functions in $S$ [10, 11], and it is easy to adapt it for functions with values in a complex Banach space $B$. We write $\mathcal{A}(S, B)$ for the space of $B$-valued holomorphic functions in $S$ that admit strong asymptotic development in $S$. The next result is due to J. A. Hernández [5] and it is based on a variant of Taylor's formula that appears in the work of Y. Haraoka [4].

**Theorem 2.8.** The following statements are equivalent:

(i) $f \in \mathcal{A}(S, B)$.

(ii) For every $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ and $T \prec S$, we have $\sup_{\boldsymbol{z}\in T} \|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\|_B < \infty$.

Taking into account the conventions adopted in the list of Notations, the asymptotic information for $f \in \mathcal{A}(S, B)$ is given by the family

$$\mathrm{TA}(f) = \left\{ f_{\boldsymbol{\alpha}_J} : \emptyset \neq J \subset \mathcal{N},\ \boldsymbol{\alpha}_J \in \mathbb{N}_0^J \right\},$$

defined for every non empty subset $J$ of $\mathcal{N}$ and every $\boldsymbol{\alpha}_J \in \mathbb{N}_0^J$ as

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J'}) = \lim_{\substack{\boldsymbol{z}_J \to \mathbf{0}_J \\ \boldsymbol{z}_J \in T_J}} D^{(\boldsymbol{\alpha}_J, \mathbf{0}_{J'})} f(\boldsymbol{z}), \quad \boldsymbol{z}_{J'} \in S_{J'}, \tag{5}$$

for every $T_J \prec S_J$; the limit is uniform on every $T_{J'} \prec S_{J'}$ whenever $J \neq \mathcal{N}$, what implies that $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J'}, B)$ ($\mathcal{A}(S_{\mathcal{N}'}, B)$ is meant to be $B$).

**Proposition 2.9** (Coherence conditions)**.** Let $f \in \mathcal{A}(S, B)$ and

$$\mathrm{TA}(f) = \left\{ f_{\boldsymbol{\alpha}_J} : \emptyset \neq J \subset \mathcal{N},\ \boldsymbol{\alpha}_J \in \mathbb{N}_0^J \right\}.$$

Then, for every pair of nonempty disjoint subsets $J$ and $L$ of $\mathcal{N}$, every $\boldsymbol{\alpha}_J \in \mathbb{N}_0^J$ and $\boldsymbol{\alpha}_L \in \mathbb{N}_0^L$, and every $T_L \prec S_L$, we have

$$\lim_{\substack{\boldsymbol{z}_L \to \mathbf{0} \\ \boldsymbol{z}_L \in T_L}} D^{(\boldsymbol{\alpha}_L, \mathbf{0}_{(J\cup L)'})} f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J'}) = f_{(\boldsymbol{\alpha}_J, \boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)'}); \tag{6}$$

the limit is uniform in each $T_{(J\cup L)'} \prec S_{(J\cup L)'}$ whenever $J \cup L \neq \mathcal{N}$.

**Definition 2.10.** We say a family

$$\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J'}, B) : \emptyset \neq J \subset \mathcal{N},\ \boldsymbol{\alpha}_J \in \mathbb{N}_0^J \,\}$$

is *coherent* if it fulfills the conditions given in (6).

**Definition 2.11.** Let $f \in \mathcal{A}(S, B)$. The *first order family* associated to $f$ is given by

$$\mathcal{B}_1(f) := \{\, f_{m_{\{j\}}} \in \mathcal{A}(S_{j'}, B) : j \in \mathcal{N},\ m \in \mathbb{N}_0 \,\} \subset \mathrm{TA}(f).$$

The first order family consists of the elements in the total family that depend on $n-1$ variables. For the sake of simplicity, we will write $f_{jm}$ instead of $f_{m_{\{j\}}}$, $j \in \mathcal{N}$, $m \in \mathbb{N}_0$. As it can be seen in [3, Section 4], knowing $\mathcal{B}_1(f)$ amounts to knowing $\mathrm{TA}(f)$.

$\mathcal{B}_1(f)$ verifies *first order coherence conditions*:

for every $L \subset \mathcal{N}$ consisting of at least two elements, every $\boldsymbol{\alpha}_L \in \mathbb{N}_0^L$, every $j, \ell \in L$ and every $T \prec S$, we have

$$\lim_{\substack{\boldsymbol{z}_{L-\{j\}} \to \mathbf{0} \\ \boldsymbol{z}_{L-\{j\}} \in S_{L-\{j\}}}} D^{(\boldsymbol{\alpha}_{L-\{j\}}, \mathbf{0}_{L'})} f_{j\alpha_j}(\boldsymbol{z}_{j'}) = \lim_{\substack{\boldsymbol{z}_{L-\{\ell\}} \to \mathbf{0} \\ \boldsymbol{z}_{L-\{\ell\}} \in S_{L-\{\ell\}}}} D^{(\boldsymbol{\alpha}_{L-\{\ell\}}, \mathbf{0}_{L'})} f_{\ell\alpha_\ell}(\boldsymbol{z}_{\ell'});$$

limits are uniform in $S_{L'}$ when $L \neq \mathcal{N}$.

In fact, there is a bijective correspondence between the set of coherent families (see Definition 2.10) and the one of coherent first order families

$$\mathcal{F}_1 = \{\, f_{jm} \in \mathcal{A}(S_{j'}, B) : j \in \mathcal{N},\ m \in \mathbb{N}_0 \,\},$$

that is, those satisfying the first order coherence conditions stated above.

According to Theorem 2.8, every element $f$ in $\mathcal{A}_{\mathbf{M},A}(S, B)$ admits strong asymptotic development in $S$, in some sense "uniform", since the limits in (5) and (6) are valid in the whole corresponding polysector.

### 2.5 Quasi-analyticity and generalizations of Watson's Lemma

We begin giving two definitions of quasi-analyticity in the classes $\mathcal{A}_{\mathbf{M}}(S)$, $S$ being a polysector in $\mathcal{R}^n$.

**Definition 2.12.** We say $\mathcal{A}_{\mathbf{M}}(S)$ is *(s) quasi-analytic* if, whenever $f \in \mathcal{A}_{\mathbf{M}}(S)$ and every element in $\mathrm{TA}(f)$ is null (or, equivalently, every function in the family $\mathcal{B}_1(f)$ is null), we have that $f$ is null in $S$. In other words, the class is (s) quasi-analytic if $\mathcal{B}_1$ is injective in the class.

We say $\mathcal{A}_{\mathbf{M}}(S)$ is *quasi-analytic* if whenever $f \in \mathcal{A}_{\mathbf{M}}(S)$ and $\mathcal{B}(f)$ is the null multi-sequence, we have that $f$ is the null function in $S$.

The authors [9] have given characterizations for both concepts in terms of the divergence of some logarithmic integrals involving the function $T_{\mathbf{M}}$ (see (3)). In the case of strongly regular sequences, we recall several generalizations of Watson's Lemma under an additional condition related to the growth index of the sequence.

**Proposition 2.13** ([9], Proposition 4.9)**.** Let $\mathbf{M}$ be strongly regular and let us suppose that

$$\sum_{n=0}^{\infty} \Big(\frac{M_n}{M_{n+1}}\Big)^{1/\gamma(\mathbf{M})} = \infty \tag{7}$$

(or, in other words, $\displaystyle\int^{\infty} \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\gamma(\mathbf{M})}}\, dr = \infty$). Given $\boldsymbol{\gamma} \in (0,\infty)^n$, define $\overline{\gamma} = \max_{1\le j\le n} \gamma_j$. The following statements are equivalent:

(i) $\overline{\gamma} \geq \gamma(\mathbf{M})$.

(ii) The class $\mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ is (s) quasi-analytic.

**Proposition 2.14** ([9], Proposition 4.14)**.** Let $\mathbf{M}$ be a strongly regular sequence that verifies (7), $\boldsymbol{\gamma} \in (0,\infty)^n$ and define $\underline{\gamma} = \min_{1\le j\le n} \gamma_j$. The following statements are equivalent:

(i) $\underline{\gamma} \geq \gamma(\mathbf{M})$.

(ii) The class $\mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ is quasi-analytic.

**Remark 2.15.** The class $\mathcal{A}_{\mathbf{M}}(S)$ is quasi-analytic (respectively, (s) quasi-analytic) if and only if the class $\mathcal{A}_{\mathbf{M}}(S,B)$ is quasi-analytic (respectively, (s) quasi-analytic) for every complex Banach space $B$. For this reason, the results on quasi-analyticity and (s) quasi-analyticity have been stated when $B=\mathbb{C}$.

# 3 Extension operators in several variables

We prove the existence of extension operators, right inverses for the maps $\mathcal{B}$ and TA. In both cases we will take the problem for functions of several variables into another equivalent one in terms of functions in one variable with values in an appropriate Banach space. In order to do this, the following result will be essential. We omit its proof, but the interested reader may find a similar result proved in detail in [6, Theorem 4.5].

**Theorem 3.1.** Let $n,m\in\mathbb{N}$, $\mathbf{M}$ be a sequence of positive real numbers, and $B$ be a complex Banach space. If we fix $A>0$ and consider (poly)sectors $S$ and $V$ in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, then we have:

(i) If $\mathbf{M}$ fulfills $(\mu)$ and $A_1>0$ is the constant involved in this property, then the map

$$\Psi_1 : \mathcal{A}_{\mathbf{M},A}(S\times V,B) \longrightarrow \mathcal{A}_{\mathbf{M},2AA_1}(S,\mathcal{A}_{\mathbf{M},2AA_1}(V,B))$$

sending each function $f\in\mathcal{A}_{\mathbf{M},A}(S\times V,B)$ to the function $f^\star=\Psi_1(f)$ given by

$$\big(f^\star(\boldsymbol{z})\big)(\boldsymbol{w}) = f(\boldsymbol{z},\boldsymbol{w}), \qquad (\boldsymbol{z},\boldsymbol{w})\in S\times V,$$

is well defined, linear and continuous. Given $f\in\mathcal{A}_{\mathbf{M},A}(S\times V,B)$, for every $\boldsymbol{\alpha}\in\mathbb{N}_0^n$, $\boldsymbol{\beta}\in\mathbb{N}_0^m$ and $(\boldsymbol{z},\boldsymbol{w})\in S\times V$ we have

$$D^{(\boldsymbol{\alpha},\boldsymbol{\beta})} f(\boldsymbol{z},\boldsymbol{w}) = D^{\boldsymbol{\beta}}\big(D^{\boldsymbol{\alpha}} f^\star(\boldsymbol{z})\big)(\boldsymbol{w}), \tag{8}$$

and so

$$\|f^\star\|^{\mathcal{A}_{\mathbf{M},2AA_1}(V,B)}_{\mathbf{M},2AA_1,S} \le \|f\|^B_{\mathbf{M},A,S\times V}.$$

(ii) If $\mathbf{M}$ fulfills $(\alpha_0)$, the map

$$\Psi_2 : \mathcal{A}_{\mathbf{M},A}(S,\mathcal{A}_{\mathbf{M},A}(V,B)) \longrightarrow \mathcal{A}_{\mathbf{M},A}(S\times V,B)$$

given by

$$\big(\Psi_2(f)\big)(\boldsymbol{w},\boldsymbol{z}) = \big(f(\boldsymbol{z})\big)(\boldsymbol{w}), \qquad (\boldsymbol{z},\boldsymbol{w})\in S\times V,$$

is well defined, linear and continuous. For $f\in\mathcal{A}_{\mathbf{M},A}(S,\mathcal{A}_{\mathbf{M},A}(V,B))$, every $\boldsymbol{\alpha}\in\mathbb{N}_0^n$, $\boldsymbol{\beta}\in\mathbb{N}_0^m$ and $(z,w)\in S\times V$ we have

$$D^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\big(\Psi_2(f)\big)(\boldsymbol{z},\boldsymbol{w}) = D^{\boldsymbol{\beta}}\big(D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\big)(\boldsymbol{w}),$$

and consequently

$$\|\Psi_2(f)\|^B_{\mathbf{M},A,S\times V} \le \|f\|^{\mathcal{A}_{\mathbf{M},A}(V,B)}_{\mathbf{M},A,S}. \tag{9}$$

The map $\mathcal{B}:\mathcal{A}_{\mathbf{M},A}(S,B)\to\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$ is linear and continuous. In the next result we obtain suitable right inverses.

**Theorem 3.2.** Let $n\in\mathbb{N}$ and $\mathbf{M}=(M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence, and let us consider $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_n)\in(0,\infty)^n$ such that $0<\gamma_j<\gamma(\mathbf{M})$ for every $j\in\{1,\ldots,n\}$. Then there exists $d\ge 1$ such that for every $A>0$ there exists a continuous linear map

$$T_{\mathbf{M},A,\gamma} : \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B) \to \mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}},B)$$

such that $\mathcal{B}\circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})=\boldsymbol{a}$ for every $\boldsymbol{a}\in\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$.

*Proof*:
We will prove this result by induction on the dimension $n \in \mathbb{N}$. For $n = 1$ the result is precisely Theorem 1.1. Let us suppose the theorem is proved for $n-1$, $n \geq 2$. Let us fix $A \in (0,\infty)$ and $\boldsymbol{a} = (a_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$. For $m \in \mathbb{N}_0$, let us consider the multi-sequence $\boldsymbol{a}_m = (a_{(m,\boldsymbol{\beta})})_{\boldsymbol{\beta}\in\mathbb{N}_0^{1'}}$. If $A_1 > 0$ is the constant involved in the property $(\mu)$ satisfied by $\mathbf{M}$, we prove that $\boldsymbol{a}_m \in \Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0^{1'}, B)$. Indeed, taking into account the definition of the norm in these spaces and that $2^{q+p}q!p! \geq (q+p)!$ for every $q, p \in \mathbb{N}_0$, we can write

$$\begin{aligned}\|a_{(m,\boldsymbol{\beta})}\|_B &\leq |\boldsymbol{a}|_{\mathbf{M},A,B} A^{|\boldsymbol{\beta}|+m}(|\boldsymbol{\beta}|+m)!M_{|\boldsymbol{\beta}|+m}\\ &\leq |\boldsymbol{a}|_{\mathbf{M},A,B} A^{|\boldsymbol{\beta}|+m}2^{|\boldsymbol{\beta}|+m}|\boldsymbol{\beta}|!m!A_1^{|\boldsymbol{\beta}|+m}M_{|\boldsymbol{\beta}|}M_m\\ &\leq |\boldsymbol{a}|_{\mathbf{M},A,B}(2AA_1)^m m!M_m(2AA_1)^{|\boldsymbol{\beta}|}|\boldsymbol{\beta}|!M_{|\boldsymbol{\beta}|},\end{aligned}$$

concluding that $\boldsymbol{a}_m \in \Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0^{1'}, B)$ and

$$|\boldsymbol{a}_m|_{\mathbf{M},2AA_1,B} \leq |a|_{\mathbf{M},A,B}(2AA_1)^m m!M_m. \tag{10}$$

By the induction hypothesis, there exist $d_{1'} \geq 1$ and $D_{1'} > 0$, and a linear continuous operator

$$\overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}} : \Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0^{1'}, B) \to \mathcal{A}_{\mathbf{M},d_{1'}2AA_1}(S_{\gamma_{1'}}, B)$$

such that

$$\mathcal{B} \circ \overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m) = \boldsymbol{a}_m \tag{11}$$

and

$$\|\overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m)\|^B_{\mathbf{M},d_{1'}2AA_1,S_{\boldsymbol{\gamma}_{1'}}} \leq D_{1'}|\boldsymbol{a}_m|_{\mathbf{M},2AA_1,B}, \qquad m \in \mathbb{N}_0.$$

By (10), we get

$$\|\overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m)\|^B_{\mathbf{M},d_{1'}2AA_1,S_{\boldsymbol{\gamma}_{1'}}} \leq D_{1'}|\boldsymbol{a}|_{\mathbf{M},A,B}(2AA_1)^m m!M_m, \qquad m \in \mathbb{N}_0,$$

so

$$\boldsymbol{b} := \left(\overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m)\right)_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0, \mathcal{A}_{\mathbf{M},d_{1'}2AA_1}(S_{\boldsymbol{\gamma}_{1'}}, B)).$$

and

$$|\boldsymbol{b}|_{\mathbf{M},2AA_1,\mathcal{A}_{\mathbf{M},d_{1'}2AA_1}(S_{\boldsymbol{\gamma}_{1'}},B)} \leq D_{1'}|\boldsymbol{a}|_{\mathbf{M},A,B}. \tag{12}$$

In addition to that, taking into account (11) we also have that for every $\boldsymbol{\beta} \in \mathbb{N}_0^{1'}$,

$$\lim_{\boldsymbol{z}_{1'}\to\boldsymbol{0}_{1'},\boldsymbol{z}_{1'}\in S_{\gamma_{1'}}} D^{\beta}(\overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m))(\boldsymbol{z}_{1'}) = a_{(m,\boldsymbol{\beta})}. \tag{13}$$

Let $B_1 = \mathcal{A}_{\mathbf{M},d_{1'}2AA_1}(S_{\gamma_{1'}}, B)$. Theorem 1.1 ensures the existence of constants $d_1 \geq 1$ and $D_1 > 0$, that only depend on $\boldsymbol{\gamma}$ and $\mathbf{M}$, and of a linear continuous map

$$\tilde{T}_{\mathbf{M},2AA_1,\gamma_1} : \Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0, B_1) \to \mathcal{A}_{\mathbf{M},d_12AA_1}(S_{\gamma_1}, B_1)$$

such that $\mathcal{B} \circ \tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b}) = \boldsymbol{b}$, it is to say, for every $m \in \mathbb{N}_0$ we have

$$\lim_{z_1\to 0,z_1\in S_{\gamma_1}} D^m(\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b}))(z_1) = \overline{T}_{\mathbf{M},2AA_1,\boldsymbol{\gamma}_{1'}}(\boldsymbol{a}_m), \tag{14}$$

and moreover

$$\|\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\|_{\mathbf{M},d_12AA_1,S_{\gamma_1},B_1} \leq D_1|\boldsymbol{b}|_{\mathbf{M},2AA_1,B_1}. \tag{15}$$

If we put $d = \max\{d_1, d_{1'}\}2A_1 \geq 1$, then it is clear that $\mathcal{A}_{\mathbf{M},d_12AA_1}(S_{\gamma_1}, B_1)$ is continuously injected in $\mathcal{A}_{\mathbf{M},dA}(S_{\gamma_1}, \mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}_{1'}}, B))$, and

$$\|f\|^{\mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}_{1'}},B)}_{\mathbf{M},dA,S_{\gamma_1}} \leq \|f\|^{B_1}_{\mathbf{M},d_12AA_1,S_{\gamma_1}}, \quad f \in \mathcal{A}_{\mathbf{M},d_12AA_1}(S_{\gamma_1}, B_1). \tag{16}$$

Let $\Psi_2$ be the linear continuous operator from $\mathcal{A}_{\mathbf{M},dA}(S_{\gamma_1},\mathcal{A}_{\mathbf{M},dA}(S_{\gamma_{1'}},B))$ to $\mathcal{A}_{\mathbf{M},dA}(S_{\gamma},B)$ described in Theorem 3.1. We define the map

$$T_{\mathbf{M},A,\boldsymbol{\gamma}}:\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)\to\mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}},B)$$

by $T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})=\Psi_2\circ\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are as before. Clearly $T_{\mathbf{M},A,\boldsymbol{\gamma}}$ is a linear map. We also have that for every $\boldsymbol{a}\in\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$, applying (9), (16), (15) and (12),

$$\begin{aligned}\|T_{\mathbf{M},A,\gamma}(\boldsymbol{a})\|_{\mathbf{M},dA,S_\gamma,B}&=\|\Psi_2\circ\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\|^{B}_{\mathbf{M},dA,S_\gamma}\\&\le\|\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\|^{\mathcal{A}_{\mathbf{M},dA}(S_{\gamma_{1'}},B)}_{\mathbf{M},dA,S_{\gamma_1}}\\&\le\|\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\|^{B_1}_{\mathbf{M},d_12AA_1,S_{\gamma_1}}\le D_1|\boldsymbol{b}|_{\mathbf{M},2AA_1,B_1}\\&\le D_1D_{1'}|\boldsymbol{a}|_{\mathbf{M},A,B},\end{aligned}$$

concluding the continuity of the operator $T_{\mathbf{M},A,\boldsymbol{\gamma}}$. Finally, we verify that this operator is a right inverse for $\mathcal{B}$. Given $a\in\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$, the family $\mathrm{TA}(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}))$ is coherent (see Proposition 2.9), and taking into account (14) and (13) we deduce that for every $\boldsymbol{\alpha}=(m,\boldsymbol{\beta})\in\mathbb{N}_0^n$,

$$\begin{aligned}\lim_{\boldsymbol{z}\to\boldsymbol{0},\boldsymbol{z}\in S_{\boldsymbol{\gamma}}}D^{\boldsymbol{\alpha}}\big(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})\big)(\boldsymbol{z})&=\lim_{\boldsymbol{z}\to 0,\boldsymbol{z}\in S_\gamma}D^{\boldsymbol{\alpha}}\big(\Psi_2\circ\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\big)(\boldsymbol{z})\\&=\lim_{\boldsymbol{z}\to\boldsymbol{0},\boldsymbol{z}\in S_{\boldsymbol{\gamma}}}D^{\boldsymbol{\beta}}\Big(D^m\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b})\big)(z_1)\Big)(\boldsymbol{z}_{1'})\\&=\lim_{\boldsymbol{z}_{1'}\to\boldsymbol{0},\boldsymbol{z}_{1'}\in S_{\boldsymbol{\gamma}_{1'}}}\ \lim_{z_1\to 0,z_1\in S_{\gamma_1}}D^{\boldsymbol{\beta}}\Big(D^m(\tilde{T}_{\mathbf{M},2AA_1,\gamma_1}(\boldsymbol{b}))(z_1)\Big)(\boldsymbol{z}_{1'})\\&=\lim_{\boldsymbol{z}_{1'}\to\boldsymbol{0},\boldsymbol{z}_{1'}\in S_{\boldsymbol{\gamma}_{1'}}}D^{\boldsymbol{\beta}}\overline{T}_{\mathbf{M},2A_1A,\gamma_1}(\boldsymbol{a}_m)(\boldsymbol{z}_{1'})=a_{(m,\boldsymbol{\beta})}=a_{\boldsymbol{\alpha}},\end{aligned}$$

as desired. □

We now consider the second problem, concerning the existence of a right inverse for the map TA. Its solution is based on a reformulation of the problem in terms of the subfamily $\mathcal{B}_1(f)\subset\mathrm{TA}(f)$.

We firstly analyze how $\mathcal{B}_1$ acts on $\mathcal{A}_{\mathbf{M},A}(S,B)$. The next definition will help in order to easily state the following result.

**Definition 3.3.** Let $\mathbf{M}=(M_p)_{p\in\mathbb{N}_0}$ be a sequence that fulfills property $(\mu)$ for a constant $A_1$, and let $A>0$. We define $\mathfrak{F}^1_{\mathbf{M},A}(S,B)$ as the set of coherent families of first order

$$\mathcal{G}=\{f_{jm}\in\mathcal{A}_{\mathbf{M},2AA_1}(S_{j'},B):j\in\mathcal{N},m\in\mathbb{N}_0\}$$

such that for every $j\in\mathcal{N}$ we have

$$\mathcal{G}_j:=(f_{jm})_{m\in\mathbb{N}_0}\in\Lambda_{\mathbf{M},2AA_1}(\mathbb{N}_0,\mathcal{A}_{\mathbf{M},2AA_1}(S_{j'},B)).$$

It is immediate to prove that, if we put

$$\nu_{\mathbf{M},A}(\mathcal{G}):=\sup_{j\in\mathcal{N}}\{|\mathcal{G}_j|_{\mathbf{M},2AA_1,\mathcal{A}_{\mathbf{M},2AA_1}(S_{j'},B)}\},\qquad\mathcal{G}\in\mathfrak{F}^1_{\mathbf{M},A}(S,B),$$

then $\big(\mathfrak{F}^1_{\mathbf{M},A}(S,B),\nu_{\mathbf{M},A}\big)$ is a Banach space.

**Proposition 3.4.** In the conditions of the previous definition, the map

$$\mathcal{B}_1:\mathcal{A}_{\mathbf{M},A}(S,B)\longrightarrow\mathfrak{F}^1_{\mathbf{M},A}(S,B)$$

is well defined, linear and continuous.

*Proof*:
Let $f \in \mathcal{A}_{\mathbf{M},A}(S,B)$ and let us fix $m \in \mathbb{N}_0$ and $j \in \mathcal{N}$. We begin recalling that

$$f_{jm}(\boldsymbol{z}_{j'}) = \lim_{z_j \to 0, z_j \in S_j} D^{(0_{j'}, m_{\{j\}})} f(\boldsymbol{z}_{j'}, z_j), \qquad \boldsymbol{z}_{j'} \in S_{j'}.$$

Due to the uniformity in the limits defining the elements in $\mathrm{TA}(f)$ we can deduce that for every $\boldsymbol{z}_{j'} \in S_{j'}$ and every $\boldsymbol{\alpha}_{j'} \in \mathbb{N}_0^{j'}$, we have

$$D^{\boldsymbol{\alpha}_{j'}} f_{jm}(\boldsymbol{z}_{j'}) = \lim_{z_j \to 0, z_j \in S_j} D^{(\boldsymbol{\alpha}_{j'}, m_{\{j\}})} f(\boldsymbol{z}_{j'}, z_j),$$

so

$$\begin{aligned} \|D^{\boldsymbol{\alpha}_{j'}} f_{jm}(\boldsymbol{z}_{j'})\|_B &\le \sup_{z_j \in S_j} \|D^{(\boldsymbol{\alpha}_{j'}, m_{\{j\}})} f(\boldsymbol{z}_{j'}, z_j)\|_B \\ &\le \|f\|^B_{\mathbf{M},A,S} A^{|\boldsymbol{\alpha}_{j'}|+m} (|\boldsymbol{\alpha}_{j'}|+m)! M_{|\boldsymbol{\alpha}_{j'}|+m} \\ &\le \|f\|^B_{\mathbf{M},A,S} A^{|\boldsymbol{\alpha}_{j'}|+m} 2^{|\boldsymbol{\alpha}_{j'}|+m} |\boldsymbol{\alpha}_{j'}|! m! A_1^{|\boldsymbol{\alpha}_{j'}|+m} M_{|\boldsymbol{\alpha}_{j'}|} M_m \\ &\le \big(\|f\|^B_{\mathbf{M},A,S} (2AA_1)^m m! M_m\big) (2AA_1)^{|\boldsymbol{\alpha}_{j'}|} |\boldsymbol{\alpha}_{j'}|! M_{|\boldsymbol{\alpha}_{j'}|}. \end{aligned}$$

Then, we see that $f_{jm} \in \mathcal{A}_{\mathbf{M},2AA_1}(S_{j'}, B)$ and

$$\|f_{jm}\|^B_{\mathbf{M},2AA_1,S_{j'}} \le \|f\|^B_{\mathbf{M},A,S} (2AA_1)^m m! M_m. \tag{17}$$

From this last inequality we now obtain that, for every $j \in \mathcal{N}$,

$$(f_{jm})_{m \in \mathbb{N}_0} \in \Lambda_{\mathbf{M},2AA_1}(\mathcal{A}_{\mathbf{M},2AA_1}(S_{j'}, B)),$$

so $\mathcal{B}_1(f) \in \mathfrak{F}^1_{\mathbf{M},A}(S,B)$. Moreover, taking into account (17),

$$|(f_{jm})_{m \in \mathbb{N}_0}|_{\mathbf{M},2AA_1,\mathcal{A}_{\mathbf{M},2AA_1}(S_{j'},B)} \le \|f\|^B_{\mathbf{M},A,S}\,, \qquad j \in \mathcal{N},$$

so that $\nu_{\mathbf{M},A}(\mathcal{B}_1(f)) \le \|f\|^B_{\mathbf{M},A,S}$. This concludes the proof. □

The next lemma supplies the information needed about the asymptotic behaviour of the function provided by the extension operator of V. Thilliez (see Theorem 1.1) when it is applied on a sequence in a space such as $\mathcal{A}_{\mathbf{M},A}(S) := \mathcal{A}_{\mathbf{M},A}(S,\mathbb{C})$. Its use will be fundamental in the proof of the extension result from the families in $\mathfrak{F}^1_{\mathbf{M},A}(S,B)$. The proof of this lemma is quite technical and lengthy, so we have decided to give only a sketch of the proof and postpone it to Section 5 not to interrupt the course of ideas.

**Lemma 3.5.** Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence, $A > 0$, $S_{\boldsymbol{\theta}}$ be a polysector in $\mathcal{R}^n$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in (0,\infty)^n$, and $S_\gamma$ be a sector in $\mathcal{R}$ with $0 < \gamma < \gamma(\mathbf{M})$. Suppose $\boldsymbol{f} = (f_p)_{p \in \mathbb{N}_0} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0, \mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}}))$ and there exists $j \in \mathcal{N}$ in such a way that for every $m, p \in \mathbb{N}_0$ we have

$$\lim_{z_j \to 0, z_j \in S_{\theta_j}} D^{me_j} f_p(\boldsymbol{z}) = 0 \quad \text{ uniformly in } S_{\boldsymbol{\theta}_{j'}}.$$

Then for every $m \in \mathbb{N}_0$ we have

$$\lim_{z_j \to 0, z_j \in S_{\theta_j}} D^{me_j} \big((T_{\mathbf{M},A,\gamma} \boldsymbol{f})(w)\big)(\boldsymbol{z}) = 0 \tag{18}$$

uniformly for $w \in S_\gamma$ and $\boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}$, where $T_{\mathbf{M},A,\gamma}$ is the extension operator given in Theorem 1.1.

We now state our second extension result.

**Theorem 3.6.** Let $\mathbf{M} = (M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence, and let us fix $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in (0,\infty)^n$ such that $0 < \theta_j < \gamma(\mathbf{M})$ for every $j \in \mathcal{N}$. Then, there exists a constant $c = c(\mathbf{M}, \boldsymbol{\theta}) \geq 1$ such that for every $A > 0$ there exists a linear continuous operator

$$U_{\mathbf{M},A,\boldsymbol{\theta}} : \mathfrak{F}^1_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B) \longrightarrow \mathcal{A}_{\mathbf{M},cA}(S_{\boldsymbol{\theta}}, B)$$

such that for every $\mathcal{G} \in \mathfrak{F}^1_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B)$ we have $\mathcal{B}_1(U_{\mathbf{M},A,\boldsymbol{\theta}}(\mathcal{G})) = \mathcal{G}$.

*Proof*:
Let $\mathcal{G} = \{f_{jm}\} \in \mathfrak{F}^1_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B)$. The proof is divided in $n$ steps, in such a way that in the $k$-th step we will obtain a function whose first order family contains the first $k$ sequences $(f_{jm})_{m\in\mathbb{N}_0}$, with $j \leq k$. The previous lemma guarantees that we are not losing in each step what was achieved in the previous ones.

According to Definition 3.3, we have

$$\mathcal{G}_1 := \{f_{1m}\}_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},2AA_1}(\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}}, B)).$$

By Theorem 1.1, we can find $c_1 = c_1(\mathbf{M}, \theta_1) \geq 1$, $C_1 = C_1(\mathbf{M}, \theta_1) > 0$ and a linear continuous operator

$$T_{\mathbf{M},2AA_1,\theta_1} : \Lambda_{\mathbf{M},2AA_1}(\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}}, B)) \to \mathcal{A}_{\mathbf{M},c_12AA_1}(S_{\theta_1}, \mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}}, B))$$

such that, if we put $H_1^{[1]\star} := T_{\mathbf{M},2AA_1,\theta_1}(\mathcal{G}_1)$, then

$$H_1^{[1]\star} \sim \sum_{m=0}^{\infty} \frac{f_{1m}}{m!} z_1^m$$

and

$$\|H_1^{[1]\star}\|^{\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}},B)}_{\mathbf{M},c_12AA_1,S_{\theta_1}} \leq C_1 |\mathcal{G}_1|_{\mathbf{M},2AA_1,\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}},B)}.$$

Taking into account that

$$\mathcal{A}_{\mathbf{M},c_12AA_1}(S_{\theta_1}, \mathcal{A}_{\mathbf{M},2AA_1}(S_{\theta_{1'}}, B)) \subseteq \mathcal{A}_{\mathbf{M},c_12AA_1}(S_{\theta_1}, \mathcal{A}_{\mathbf{M},c_12AA_1}(S_{\theta_{1'}}, B))$$

(with the correspondent inequality for the norms) and the second item in Theorem 3.1, we can say that the function $H^{[1]} : S_{\boldsymbol{\theta}} \to B$ given by

$$H^{[1]}(\boldsymbol{z}) := H_1^{[1]\star}(z_1)(\boldsymbol{z}_{1'}), \qquad \boldsymbol{z} = (z_1, \boldsymbol{z}_{1'}) \in S_{\boldsymbol{\theta}},$$

belongs to $\mathcal{A}_{\mathbf{M},c_12AA_1}(S_{\boldsymbol{\theta}}, B)$ and, in addition,

$$\|H^{[1]}\|^B_{\mathbf{M},c_12AA_1,S_{\boldsymbol{\theta}}} \leq \|H_1^{[1]\star}\|^{\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\theta}_{1'}},B)}_{\mathbf{M},c_12AA_1,S_{\theta_1}}.$$

Let $\mathcal{B}_1(H^{[1]}) = \{h^{[1]}_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$. For every $\boldsymbol{z}_{1'} \in S_{\boldsymbol{\theta}_{1'}}$ we have

$$\begin{aligned} h^{[1]}_{1m}(\boldsymbol{z}_{1'}) &= \lim_{z_1\to 0, z_1\in S_{\theta_1}} D^{me_1} H^{[1]}(\boldsymbol{z}) \\ &= \lim_{z_1\to 0, z_1\in S_{\theta_1}} (H_1^{[1]\star})^{(m)}(z_1)(\boldsymbol{z}_{1'}) = f_{1m}(\boldsymbol{z}_{1'}). \end{aligned}$$

This concludes the first step of the proof, and we proceed with the second one. Let $H_2^{[1]\star}$ be the function given by

$$H_2^{[1]\star}(z_2)(\boldsymbol{z}_{2'}) := H^{[1]}(z_2, \boldsymbol{z}_{2'}), \qquad z_2 \in S_{\theta_2},\ \boldsymbol{z}_{2'} \in S_{\boldsymbol{\theta}_{2'}}.$$

From (i) in Theorem 3.1, we have

$$H_2^{[1]\star} \in \mathcal{A}_{\mathbf{M},c_1(2A_1)^2A}(S_{\theta_2}, \mathcal{A}_{\mathbf{M},c_1(2A_1)^2A}(S_{\boldsymbol{\theta}_{2'}}, B)).$$

We will write

$$H_2^{[1]\star} \sim \sum_{m=0}^{\infty} \frac{h_{2m}^{[1]}}{m!} z_2^m, \quad \text{and} \quad B_2 := \mathcal{A}_{\mathbf{M},c_1(2A_1)^2A}(S_{\boldsymbol{\theta}_{2'}}, B).$$

As $H^{[1]} \in \mathcal{A}_{\mathbf{M},c_1 2A_1 A}(S_{\boldsymbol{\theta}}, B)$, Proposition 3.4 tells us that

$$(h_{2m}^{[1]})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},c_1(2A_1)^2A}(B_2),$$

whilst Definition 3.3 allows us to write

$$\mathcal{G}_2 := (f_{2m})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},2A_1A}(\mathcal{A}_{\mathbf{M},2A_1A}(S_{\boldsymbol{\theta}_{2'}}, B)).$$

So, $(f_{2m}-h_{2m}^{[1]})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},c_1(2A_1)^2A}(B_2)$. Applying again Theorem 1.1, we find constants $c_2 = c_2(\mathbf{M}, \theta_2) \geq 1$, $C_2 = C_2(\mathbf{M}, \theta_2) > 0$ and a linear continuous operator

$$T_{\mathbf{M},c_1(2A_1)^2A,\theta_2} : \Lambda_{\mathbf{M},c_1(2A_1)^2A}(B_2) \longrightarrow \mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\theta_2}, B_2)$$

such that, if we define

$$H_2^{[2]\star} := T_{\mathbf{M},c_1(2A_1)^2A,\theta_2}\big((f_{2m} - h_{2m}^{[1]})_{m\in\mathbb{N}_0}\big),$$

then

$$H_2^{[2]\star} \sim \sum_{m=0}^{\infty} \frac{f_{2m} - h_{2m}^{[1]}}{m!} z_2^m \tag{19}$$

and

$$\|H_2^{[2]\star}\|^{B_2}_{\mathbf{M},c_2c_1(2A_1)^2A,S_{\theta_2}} \leq C_2|(f_{2m} - h_{2m}^{[1]})_{m\in\mathbb{N}_0}|_{\mathbf{M},c_1(2A_1)^2A,B_2}.$$

As in the first step, it is clear that

$$\mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\theta_2}, B_2) \subseteq \mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\theta_2}, \mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\boldsymbol{\theta}_{2'}}, B)),$$

so $H_2^{[2]\star}$ also belongs to the second of these spaces. Part (ii) in Theorem 3.1 ensures that the function $H^{[2]} : S_{\boldsymbol{\theta}} \to B$ given by

$$H^{[2]}(\boldsymbol{z}) := H_2^{[2]\star}(z_2)(\boldsymbol{z}_{2'}), \qquad \boldsymbol{z} = (z_2, \boldsymbol{z}_{2'}) \in S_{\boldsymbol{\theta}},$$

belongs to $\mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\boldsymbol{\theta}}, B)$ and

$$\|H^{[2]}\|^{B}_{\mathbf{M},c_2c_1(2A_1)^2A,S_{\boldsymbol{\theta}}} \leq \|H_2^{[2]\star}\|^{B_2}_{\mathbf{M},c_1(2A_1)^2A,S_{\theta_2}}.$$

We will write $\mathcal{B}_1(H^{[2]}) = \{h_{jm}^{[2]} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$. We now compute the elements in this family for $j = 1$ and $j = 2$. We note that, for the sake of brevity, we denote by $D^{m\boldsymbol{e}_k}h$, $k \in \{2, \ldots, n\}$, the $m$-th derivative with respect to $z_k$ of a function $h$ with variables $\boldsymbol{z}_{1'} = (z_2, \ldots, z_n)$, although this does not perfectly agree with the definition we made for $\boldsymbol{e}_k$. For $j = 1$, due to the coherence conditions of the families $\mathcal{G}$ and $\mathcal{B}_1(H^{[1]})$ we have for all $m, k \in \mathbb{N}_0$,

$$\lim_{z_1\to 0, z_1\in S_{\theta_1}} D^{me_1}(f_{2k} - h_{2k}^{[1]})(\boldsymbol{z}_{2'}) = \lim_{z_2\to 0, z_2\in S_{\theta_2}} D^{ke_2}(f_{1m} - h_{1m}^{[1]})(\boldsymbol{z}_{1'}) = 0,$$

uniformly in $S_{\boldsymbol{\theta}_{\{1,2\}'}}$. So, we can apply Lemma 3.5 to guarantee that for every $m \in \mathbb{N}_0$, we have

$$\lim_{z_1\to 0, z_1\in S_{\theta_1}} \big(H_2^{[2]\star}\big)^{(m)}(z_1)(\boldsymbol{z}_{1'}) = 0 \quad \text{uniformly in } S_{\boldsymbol{\theta}_{1'}},$$

and consequently, by Theorem 3.1 we deduce for every $\boldsymbol{z}_{1'} \in S_{\boldsymbol{\theta}_{1'}}$ that

$$h_{1m}^{[2]}(\boldsymbol{z}_{1'}) = \lim_{z_1\to 0, z_1\in S_{\theta_1}} D^{me_1}H^{[2]}(z1, \boldsymbol{z}_{1'}) = \lim_{z_1\to 0, z_1\in S_{\theta_1}} \big(H_2^{[2]\star}\big)^{(m)}(z_1)(\boldsymbol{z}_{1'}) = 0.$$

On the other hand, taking (19) into account, for every $\boldsymbol{z}_{2'} \in S_{\boldsymbol{\theta}_{2'}}$ we have

$$\begin{aligned} h^{[2]}_{2m}(\boldsymbol{z}_{2'}) &= \lim_{z_2\to 0, z_2\in S_{\theta_2}} D^{me_2} H^{[2]}(z_2, \boldsymbol{z}_{2'}) \\ &= \lim_{z_2\to 0, z_2\in S_{\theta_2}} (H^{[2]\star}_2)^{(m)}(z_2)(\boldsymbol{z}_{2'}) = (f_{2m} - h^{[1]}_{2m})(\boldsymbol{z}_{2'}). \end{aligned}$$

We define the function $F^{[2]} := H^{[1]} + H^{[2]} \in \mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^2A}(S_{\boldsymbol{\theta}}, B)$, and put $\mathcal{B}_1(F^{[2]}) = \{f^{[2]}_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$. According to the previous calculations, for every $m \in \mathbb{N}_0$ we have

$$\begin{aligned} &f^{[2]}_{1m} = h^{[1]}_{1m} + h^{[2]}_{1m} = h^{[1]}_{1m} = f_{1m}, \\ \text{and} \quad &f^{[2]}_{2m} = h^{[1]}_{2m} + h^{[2]}_{2m} = h^{[1]}_{2m} + f_{2m} - h^{[1]}_{2m} = f_{2m}, \end{aligned}$$

so the second step is completed. If we had $n = 2$, the proof would have already concluded. If $n \geq 3$, we will sketch the next step to clarify how the argument works.

We will write from now on

$$B_3 := \mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^3A}(S_{\boldsymbol{\theta}_{3'}}, B).$$

The function $F^{[2]\star}_3$, given by

$$F^{[2]\star}_3(z_3)(\boldsymbol{z}_{3'}) := F^{[2]}(z_3, \boldsymbol{z}_{3'}), \quad z_3 \in S_{\theta_3},\ \boldsymbol{z}_{3'} \in S_{\boldsymbol{\theta}_{3'}},$$

is, according to (i) in Theorem 3.1, an element in $\mathcal{A}_{\mathbf{M},c_2c_1(2A_1)^3A}(S_{\theta_3}, B_3)$, and

$$F^{[2]\star}_3 \sim \sum_{m=0}^{\infty} \frac{f^{[2]}_{3m}}{m!} z_3^m.$$

Proposition 3.4 ensures that $(f^{[2]}_{3m})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},c_2c_1(2A_1)^3A}(B_3)$, and Definition 3.3 tells us that

$$\mathcal{G}_3 := (f_{3m})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},2A_1A}(\mathcal{A}_{\mathbf{M},2A_1A}(S_{\boldsymbol{\theta}_{3'}}, B)),$$

so that $(f_{3m} - f^{[2]}_{3m})_{m\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},c_2c_1(2A_1)^3A}(B_3)$. Theorem 1.1 assures the existence of constants $c_3 = c_3(\mathbf{M}, \theta_3) \geq 1$, $C_3 = C_3(\mathbf{M}, \theta_3) > 0$ and a linear continuous operator

$$T_{\mathbf{M},c_2c_1(2A_1)^3A,\theta_3} : \Lambda_{\mathbf{M},c_2c_1(2A_1)^3A}(B_3) \longrightarrow \mathcal{A}_{\mathbf{M},c_3c_2c_1(2A_1)^3A}(S_{\theta_3}, B_3)$$

such that, if we put

$$H^{[3]\star}_3 := T_{\mathbf{M},c_2c_1(2A_1)^3A,\theta_3}\big((f_{3m} - f^{[2]}_{3m})_{m\in\mathbb{N}_0}\big),$$

then

$$H^{[3]\star}_3 \sim \sum_{m=0}^{\infty} \frac{f_{3m} - f^{[2]}_{3m}}{m!} z_3^m \tag{20}$$

and

$$\|H^{[3]\star}_3\|^{B_3}_{\mathbf{M},c_3c_2c_1(2A_1)^3A,S_{\theta_3}} \leq C_3 |(f_{3m} - f^{[2]}_{3m})_{m\in\mathbb{N}_0}|_{\mathbf{M},c_2c_1(2A_1)^3A,B_3}.$$

As in the previous step we deduce that the function $H^{[3]} : S_\theta \to B$ given by

$$H^{[3]}(\boldsymbol{z}) := H^{[3]\star}_3(z_3)(\boldsymbol{z}_{3'}), \qquad \boldsymbol{z} = (z_3, \boldsymbol{z}_{3'}) \in S_{\boldsymbol{\theta}},$$

belongs to $\mathcal{A}_{\mathbf{M},c_3c_2c_1(2A_1)^3A}(S_{\boldsymbol{\theta}}, B)$ and

$$\|H^{[3]}\|^{B}_{\mathbf{M},c_3c_2c_1(2A_1)^3A,S_{\boldsymbol{\theta}}} \leq \|H^{[3]\star}_3\|^{B_3}_{\mathbf{M},c_2c_1(2A_1)^3A,S_{\theta_3}}.$$

If we put $\mathcal{B}_1(H^{[3]}) = \{h^{[3]}_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$, following a similar argument to the one in the second step we have:

(1) Since the families $\mathcal{G}$ and $\mathcal{B}_1(H^{[2]})$ are coherent, by Lemma 3.5 we get

$$h^{[3]}_{jm}(\boldsymbol{z}_{j'}) = 0, \qquad \boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}, \quad j = 1, 2.$$

(2) According to (20), for every $\boldsymbol{z}_{3'} \in S_{\boldsymbol{\theta}_{3'}}$

$$h^{[3]}_{3m}(\boldsymbol{z}_{3'}) = (f_{3m} - f^{[2]}_{3m})(\boldsymbol{z}_{3'}), \qquad m \in \mathbb{N}_0.$$

So, the function $F^{[3]} := F^{[2]} + H^{[3]} \in \mathcal{A}_{\mathbf{M},c_3c_2c_1(2A_1)^3A}(S_{\boldsymbol{\theta}}, B)$ verifies that, if we put $\mathcal{B}_1(F^{[3]}) = \{f^{[3]}_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$, then for every $m \in \mathbb{N}_0$ and for $j \in \{1, 2, 3\}$ we have $f^{[3]}_{jm} = f_{jm}$, as desired.

After $n$ steps we obtain a function

$$F = F^{[n]} = H^{[1]} + \ldots + H^{[n]} =: U_{\mathbf{M},A,\boldsymbol{\theta}}(\mathcal{G})$$

that solves the problem. In fact, the construction tells us that $U_{\mathbf{M},A,\theta}$ is linear and sends $\mathfrak{F}^1_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B)$ into $\mathcal{A}_{\mathbf{M},cA}(S_{\boldsymbol{\theta}}, B)$, where

$$c = c(\mathbf{M}, \boldsymbol{\theta}) := c_n c_{n-1} \ldots c_2 c_1 (2A_1)^n \geq 1.$$

In addition to that, for every $\mathcal{G} \in \mathfrak{F}^1_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B)$ we have $\mathcal{B}_1(U_{\mathbf{M},A,\theta}(\mathcal{G})) = \mathcal{G}$. In order to obtain the continuity of this map, we observe that

$$\begin{aligned}\|H^{[1]}\|^B_{\mathbf{M},cA,S_\theta} &\leq \|H^{[1]}\|^B_{\mathbf{M},c_1 2A_1 A,S_{\boldsymbol{\theta}}} \leq \|H^{[1]\star}_1\|^{\mathcal{A}_{\mathbf{M},2BA}(S_{\boldsymbol{\theta}_{1'}},B)}_{\mathbf{M},c_1 2A_1 A,S_{\theta_1}} \\ &\leq C_1 |\mathcal{G}_1|_{\mathbf{M},2A_1A,S_{\theta_1},\mathcal{A}_{\mathbf{M},c_1 2A_1 A}(S_{\boldsymbol{\theta}_{1'}},B)} \leq C_1 \nu_{\mathbf{M},A}(\mathcal{G});\end{aligned}$$

$$\begin{aligned}\|H^{[2]}\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} &\leq \|H^{[2]}\|^B_{\mathbf{M},c_2c_1(2A_1)^2A,S_{\boldsymbol{\theta}}} \leq \|H^{[2]\star}_2\|^{B_2}_{\mathbf{M},c_1(2A_1)^2A,S_{\theta_2}} \\ &\leq C_2 |(f_{2m} - h^{[1]}_{2m})_{m\in\mathbb{N}_0}|_{\mathbf{M},c_1(2A_1)^2A,B_2} \\ &\leq C_2\big(|(f_{2m})_{m\in\mathbb{N}_0}|_{\mathbf{M},(2A_1)^2A,B_2} + |(h^{[1]}_{2m})_{m\in\mathbb{N}_0}|_{\mathbf{M},(2A_1)^2A,B_2}\big) \\ &\leq C_2\big(|(f_{2m})_{m\in\mathbb{N}_0}|_{\mathbf{M},2A_1A,B_2} + |\mathcal{J}(H^{[1]\star}_2)|_{\mathbf{M},c_1(2A_1)^2A,B_2}\big) \\ &\leq C_2(\nu_{\mathbf{M},A}(\mathcal{G}) + \|H^{[1]}\|^B_{\mathbf{M},c_1 2A_1A,S_{\boldsymbol{\theta}}}) \leq C_2(1 + C_1)\nu_{\mathbf{M},A}(\mathcal{G}).\end{aligned}$$

After $j$ steps, we can also prove that

$$\|H^{[j]}\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} \leq C_j \prod_{k=1}^{j-1} (1 + C_k)\nu_{\mathbf{M},A}(\mathcal{G}).$$

So,

$$\begin{aligned}\|F^{[j]}\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} &= \Big\|\sum_{k=1}^{j} H^{[k]}\Big\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} \leq \sum_{k=1}^{j} \|H^{[k]}\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} \\ &\leq \sum_{k=1}^{j} C_k \prod_{\ell=1}^{k-1} (1 + C_\ell)\nu_{\mathbf{M},A}(\mathcal{G}) = \Big(\prod_{k=1}^{j}(1 + C_k) - 1\Big)\nu_{\mathbf{M},A}(\mathcal{G}).\end{aligned}$$

In particular,

$$\|U_{\mathbf{M},A,\theta}(\mathcal{G})\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} = \|F^{[n]}\|^B_{\mathbf{M},cA,S_{\boldsymbol{\theta}}} \leq \underbrace{\Big(\prod_{k=1}^{n}(1 + C_k) - 1\Big)}_{C=C(\mathbf{M},\boldsymbol{\theta})>0} \nu_{\mathbf{M},A}(\mathcal{G}),$$

what concludes the proof. □

Our next aim is to obtain some results about the necessity of the condition $\overline{\gamma} < \gamma(\mathbf{M})$ for the existence of the extension operators. We only work under hypothesis (7), so conclusions are restricted to certain ultraholomorphic classes that, nevertheless, include Gevrey classes, among others. The proof of (iv)⇒(i) in the following result, in the one variable case, is an adaptation of the one given by J. Schmets and M. Valdivia for a similar result related to Gevrey classes [16, Theorem 5.11], which is so generalized.

**Theorem 3.7.** Let $\mathbf{M}$ be a strongly regular sequence that satisfies (7), $n \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (0,\infty)^n$. The following statements are equivalent:

(i) $\overline{\gamma} < \gamma(\mathbf{M})$.

(ii) There exists $d \geq 1$ such that for every $A > 0$ there is a linear continuous operator

$$T_{\mathbf{M},A,\boldsymbol{\gamma}} : \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n) \to \mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}})$$

such that $\mathcal{B} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}}$ is the identity map in $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n)$.

(iii) The Borel map $\mathcal{B} : \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}}) \to \Lambda_{\mathbf{M}}(\mathbb{N}_0^n)$ is surjective.

(iv) There exists a function $f \in \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ such that for every $j \in \mathcal{N}$ and every $m \in \mathbb{N}_0$ we have $D^{m\boldsymbol{e}_j} f(\mathbf{0}) = \delta_{1,m}$ (where $\delta_{1,m}$ stands for Kronecker's delta).

*Proof*:
(i)⇒(ii) It is Theorem 3.2.
(ii)⇒(iii) Given $\boldsymbol{\lambda} \in \Lambda_{\mathbf{M}}(\mathbb{N}_0^n)$, there exists $A > 0$ such that $\boldsymbol{\lambda} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n)$. So, $T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{\lambda}) \in \mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}}) \subset \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$, and we have $\mathcal{B}\big(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{\lambda})\big) = \boldsymbol{\lambda}$, so that $\mathcal{B}$ is surjective.
(iii)⇒(iv) Let us consider the family of complex numbers $\boldsymbol{\lambda} = (\lambda_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}$ given by

$$\lambda_{\boldsymbol{\alpha}} = 1 \quad \text{if } \boldsymbol{\alpha} = \boldsymbol{e}_j \text{ for some } j \in \mathcal{N}; \quad \lambda_{\boldsymbol{\alpha}} = 0 \quad \text{otherwise.}$$

It is obvious that $\boldsymbol{\lambda} \in \Lambda_{\mathbf{M}}(\mathbb{N}_0^n)$, and according to (iii), there exists $f \in \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ such that $\tilde{\mathcal{B}}(f) = \boldsymbol{\lambda}$, so that, in particular, for every $j \in \mathcal{N}$ and every $m \in \mathbb{N}_0$, $D^{m\boldsymbol{e}_j} f(\mathbf{0}) = \delta_{1,m}$.
(iv)⇒(i) We will first suppose $n = 1$. In this case, $\boldsymbol{\gamma}$ is a constant $\gamma \in (0,\infty)$, and $\overline{\gamma} = \gamma$. Let $f$ be the map in (iv) and let $A > 0$ be such that $f \in \mathcal{A}_{\mathbf{M},A}(S_\gamma)$. $f$ is bounded in $S_\gamma$, so $\phi : S_\gamma \to \mathbb{C}$ given by $\phi(z) := f(z) - z$ is not identically 0 in $S_\gamma$. Applying Taylor's formula at 0 we conclude the existence of a constant $C > 0$ such that if $z \in S_\gamma$ with $|z| \leq 1$, and $p \in \mathbb{N}_0$,

$$\begin{aligned} |\phi(z)| &= \Big| \int_0^z \frac{w^p}{p!} \phi^{(p)}(w) dw \Big| \\ &\leq |z|^p C A^p M_p. \end{aligned}$$

So, the holomorphic map $\Psi : \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \to \mathbb{C}$ given by $\Psi(u) = \phi(1/u^\gamma)$ is not identically 0 and

$$|\Psi(u)| = \Big|\phi(\frac{1}{u^\gamma})\Big| \leq \frac{CA^p M_p}{|u|^{\gamma p}},$$

for every $p \in \mathbb{N}_0, \mathrm{Re}(u) \geq 1$. Applying Theorem 2.4.III in [12] and taking into account that

$$\lim_{p\to\infty} \frac{M_{p+1}}{M_p} = \lim_{p\to\infty} M_p^{1/p} = \infty,$$

we have

$$\sum_{p=0}^{\infty} \Big(\frac{M_p}{M_{p+1}}\Big)^{1/\gamma} < \infty,$$

what, according to (7), would not be possible if we had $\gamma \geq \gamma(\mathbf{M})$.

If $n > 1$, for a map $f$ as in the hypothesis, we can consider the element $f_{\mathbf{0}_{j'}} \in \mathrm{TA}(f)$, $j \in \mathcal{N}$, given by (see (5))

$$f_{\mathbf{0}_{j'}}(z_j) = \lim_{\boldsymbol{z}_{j'}\to\mathbf{0}_{j'},\ \boldsymbol{z}_{j'}\in S_{\boldsymbol{\gamma}_{j'}}} f(z_j, \boldsymbol{z}_{j'}), \qquad z_j \in S_{\gamma_j}.$$

Since the limits defining the elements in $\mathrm{TA}(f)$ are uniform, for every $z_j \in S_{\gamma_j}$ and every $m \in \mathbb{N}_0$ we have

$$f_{\mathbf{0}_{j'}}^{(m)}(z_j) = \lim_{\boldsymbol{z}_{j'}\to\mathbf{0}_{j'},\ \boldsymbol{z}_{j'}\in S_{\boldsymbol{\gamma}_{j'}}} D^{m\boldsymbol{e}_j} f(z_j, \boldsymbol{z}_{j'}),$$

so, if $z_j \in S_{\gamma_j}$ it holds

$$|f_{\mathbf{0}_{j'}}^{(m)}(z_j)| \leq \sup_{\boldsymbol{z} \in S_{\boldsymbol{\gamma}}} |D^{m\boldsymbol{e}_j} f(z_j, \boldsymbol{z}_{j'})| \leq \|f\|_{\mathbf{M},A,S_{\boldsymbol{\gamma}}} A^m m! M_m.$$

Then, $f_{\mathbf{0}_{j'}} \in \mathcal{A}_{\mathbf{M},A}(S_{\gamma_j})$, and for every $m \in \mathbb{N}_0$ we have

$$f_{\mathbf{0}_{j'}}^{(m)}(0) := \lim_{z_j \to 0,\ z_j \in S_{\gamma_j}} f_{\mathbf{0}_{j'}}^{(m)}(z_j) = \lim_{\boldsymbol{z} \to \mathbf{0}, \boldsymbol{z} \in S_{\boldsymbol{\gamma}}} D^{m\boldsymbol{e}_j} f(\boldsymbol{z}) = \delta_{1,m}.$$

By applying the first part in this item to each map $f_{\mathbf{0}_{j'}}$, we deduce that $\gamma_j < \gamma(\mathbf{M})$ for every $j$, it is to say, $\overline{\gamma} < \gamma(\mathbf{M})$, as desired. □

**Remark 3.8.** Under the conditions in the previous theorem, and in the one variable setting, Proposition 2.14 supplies another equivalent statement to (i)-(iv): the class $\mathcal{A}_{\mathbf{M}}(S_\gamma)$ is not quasi-analytic.

It is also worth saying that this theorem can be applied to maps with values in a complex Banach space $B$, replacing (iv) in the following way:

(iv') There exists a map $f \in \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}}, B)$ such that for every $j \in \mathcal{N}$ and every $m \in \mathbb{N}_0 \setminus \{1\}$ we have $D^{m\boldsymbol{e}_j} f(\mathbf{0}) = 0$, while $D^{\boldsymbol{e}_j} f(\mathbf{0}) \neq 0$ for every $j \in \mathcal{N}$.

The only step that changes in the proof is (iv')⇒(i): for $n = 1$, it is sufficient to choose, by Hahn-Banach theorem, a linear continuous functional $\varphi : B \to \mathbb{C}$ such that $\varphi(f'(0)) = 1$, and observe that the map $\varphi \circ f$ is an element in $\mathcal{A}_{\mathbf{M}}(S_\gamma)$ satisfying conditions in (iv), so we get (i). In the several variables case the argument is analogous.

To end this section we establish an application of the result we have just mentioned.

**Proposition 3.9.** Let $\mathbf{M}$ be a strongly regular sequence satisfying (7), $n \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (0,\infty)^n$. Let us suppose that for a map $f \in \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$, with

$$\mathcal{B}_1(f) = \{f_{jm} : j \in \mathcal{N},\ m \in \mathbb{N}_0\},$$

there exists $j \in \mathcal{N}$ such that $f_{jm} \equiv 0$ if $m \in \mathbb{N}_0 \setminus \{1\}$, and $f_{j1} \not\equiv 0$. Then, we have $\gamma_j < \gamma(\mathbf{M})$.

*Proof*:
Let $A > 0$ be such that $f \in \mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}})$, and let $A_1$ be the constant involved in the property $(\mu)$ for the sequence $\mathbf{M}$. We will write $B$ for the Banach space $\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\gamma}_{j'}})$, where $j$ is the one in the hypothesis. According to the statement (i) in Theorem 3.1, we can consider the map $f_j^* : S_{\gamma_j} \to B$ given by

$$(f_j^*(z_j))(\boldsymbol{z}_{j'}) = f(z_j, \boldsymbol{z}_{j'}), \qquad z_j \in S_{\gamma_j},\ \boldsymbol{z}_{j'} \in S_{\boldsymbol{\gamma}_{j'}}.$$

We know that $f_j^* \in \mathcal{A}_{\mathbf{M},2AA_1}(S_{\gamma_j}, B)$, and that for every $m \in \mathbb{N}_0$ we have $(f_j^*)^{(m)}(0) = f_{jm}$. According to the hypothesis on $f_{jm}$, we are allowed to apply the version of Theorem 3.7 commented on in the previous remark, and we conclude. □

# 4 Rigidity of the extension operators

In this section we will discuss some rigidity results for the extension operators built in Section 3. These problems are inspired by the ones posed by V. Thilliez in [17]. Let $n \in \mathbb{N}$, $B$ be a complex Banach space, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence and $A > 0$. We will begin defining the spaces that will be considered in the study of the operators in Theorem 3.2. Our aim is to determine annihilation conditions on the interpolating functions which ensure the interpolated data are null.

We define the set $\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ as that consisting of the multi-sequences $\boldsymbol{a} = (a_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}_0^n}$ of elements in $B$ such that

$$\lim_{|\boldsymbol{\alpha}| \to \infty} \frac{\|a_{\boldsymbol{\alpha}}\|_B}{A^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|}} = 0.$$

It is clear that $\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B)\subseteq\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$ and $(\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B),|\cdot|_{\mathbf{M},A,B})$ is a Banach space, since it is closed in the Banach space $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n,B)$.

Let us consider $\boldsymbol{\gamma}\in(0,\infty)^n$ such that $\overline{\gamma}<\gamma(\mathbf{M})$, and let $d=d(\mathbf{M},\boldsymbol{\gamma})$ be the constant obtained in Theorem 3.2. We define the set $\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ as that consisting of the functions $f\in\mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}},B)$ such that $\mathcal{B}(f)\in\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B)$, and consider the norm

$$\|f\|^{\circ B}_{\mathbf{M},A,S_{\boldsymbol{\gamma}}}:=\|f\|^{B}_{\mathbf{M},dA,S_{\boldsymbol{\gamma}}}+|\mathcal{B}(f)|_{\mathbf{M},A,B},\qquad f\in\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B).$$

$(\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B),\|\cdot\|^{\circ B}_{\mathbf{M},A,S_{\boldsymbol{\gamma}}})$ is a complex Banach space, since $\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ is closed in $\mathcal{A}_{\mathbf{M},dA}(S_{\boldsymbol{\gamma}},B)$.

We will next consider the subspace $\mathcal{K}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ consisting of the maps $f\in\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ such that $\mathcal{B}(f)\equiv 0$. Since $\mathcal{K}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ is closed in $\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$, the quotient space

$$\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B):=\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)/\mathcal{K}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$$

is a Banach space with the norm

$$\begin{aligned}\nu^{\circ}_{\mathbf{M},A}(\dot f)&:=\inf_{g\in\mathcal{K}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)}\|f+g\|^{\circ B}_{\mathbf{M},A,S_{\boldsymbol{\gamma}}}\\&=\inf_{g\in\mathcal{K}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)}\|f+g\|^{B}_{\mathbf{M},dA,S_{\boldsymbol{\gamma}}}+|\mathcal{B}(f)|_{\mathbf{M},A,B},\quad f\in\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B).\end{aligned}$$

Let $\pi_{\mathbf{M},A}:\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)\to\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ be the quotient map. It is clear that $\mathcal{B}$ induces the isomorphism

$$\dot{\mathcal{B}}:\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)\to\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B),$$

with inverse $\pi_{\mathbf{M},A}\circ T_{\mathbf{M},A,\boldsymbol{\gamma}}$.

For every $\boldsymbol{a}\in\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B)$ we have

$$\begin{aligned}\nu^{\circ}_{\mathbf{M},A}(\dot{\mathcal{B}}^{-1}(\boldsymbol{a}))&=\nu^{\circ}_{\mathbf{M},A}(\pi_{\mathbf{M},A}\circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}))\le\|T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})\|^{\circ B}_{\mathbf{M},A,S_{\boldsymbol{\gamma}}}\\&=\|T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})\|^{B}_{\mathbf{M},A,S_{\boldsymbol{\gamma}}}+|\mathcal{B}(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}))|_{\mathbf{M},A,B}\\&\le C|\boldsymbol{a}|_{\mathbf{M},A,B}+|\boldsymbol{a}|_{\mathbf{M},A,B}=(1+C)|\boldsymbol{a}|_{\mathbf{M},A,B},\end{aligned}$$

$C$ being the norm of $T_{\mathbf{M},A,\boldsymbol{\gamma}}$. Therefore,

$$\left\|\dot{\mathcal{B}}^{-1}\right\|\le 1+C.$$

We can also define the map

$$P_A:\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)\to\mathcal{A}^{\circ}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$$

given by $P_A=T_{\mathbf{M},A,\boldsymbol{\gamma}}\circ\mathcal{B}$; $P_A$ is linear and continuous.

Let us suppose that for every $\boldsymbol{\alpha}\in\mathbb{N}_0^n$ a point in $S_{\boldsymbol{\gamma}}$, $z_{\boldsymbol{\alpha}}=(z^{(1)}_{\boldsymbol{\alpha}},\dots,z^{(n)}_{\boldsymbol{\alpha}})$, is chosen, and consider the map

$$\dot{\mathcal{D}}:\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)\to B^{\mathbb{N}_0^n}$$

given by

$$\dot{\mathcal{D}}(\dot f)=(D^{\boldsymbol{\alpha}}(P_Af)(z_{\boldsymbol{\alpha}}))_{\boldsymbol{\alpha}\in\mathbb{N}_0^n},\qquad \dot f\in\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B).$$

The map $\dot{\mathcal{D}}$ is well defined: If $\dot f_1,\dot f_2\in\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ and $\dot f_1=\dot f_2$, then $\mathcal{B}(f_1-f_2)=\mathcal{B}(f_1)-\mathcal{B}(f_2)\equiv 0$, being $f_1$ and $f_2$ elements in $\dot f_1$ and $\dot f_2$, respectively; so, $P_A(\dot f_1)=P_A(\dot f_2)$.

Under certain conditions on the points $z_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha}\in\mathbb{N}_0^n$, we will prove that the map $\dot{\mathcal{D}}$ is "close enough" to $\dot{\mathcal{B}}$.

**Lemma 4.1.** Let $n\in\mathbb{N}$, $\mathbf{M}=(M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence, $A>0$, $\boldsymbol{\gamma}\in(0,\infty)^n$ such that $\overline{\gamma}<\gamma(\mathbf{M})$, and $(\boldsymbol{z}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}$ be a multi-sequence of elements in $S_{\boldsymbol{\gamma}}$, $\boldsymbol{z}_{\boldsymbol{\alpha}}=(z^{(1)}_{\boldsymbol{\alpha}},\dots,z^{(n)}_{\boldsymbol{\alpha}})$. Let us suppose there exists a constant $k\in(0,1)$ such that

$$CAM_1(|\boldsymbol{\alpha}|+1)(dA_1)^{|\boldsymbol{\alpha}|+1}\sum_{j=1}^{n}|z^{(j)}_{\boldsymbol{\alpha}}|\le\frac{k}{(1+C)},\qquad\text{for every }\boldsymbol{\alpha}\in\mathbb{N}_0^n,\tag{21}$$

where $d \geq 1$ and $C > 0$ are the constants mentioned before, and $A_1$ is the constant that appears in the property $(\mu)$ for $\mathbf{M}$. Then, the image of $\dot{\mathcal{D}}$ is contained in $\Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ and $\dot{\mathcal{D}}$ admits a continuous inverse.

*Proof*:
Let $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ and $\dot{f} \in \mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}}, B)$. The $\boldsymbol{\alpha}$-element in $\dot{\mathcal{B}}(\dot{f})$ is $D^{(\boldsymbol{\alpha})}(P_A f)(\mathbf{0})$, so

$$\begin{aligned} d_{\boldsymbol{\alpha}} &:= \left\| D^{(\boldsymbol{\alpha})}(P_A f)(\boldsymbol{z}_{\boldsymbol{\alpha}}) - D^{\boldsymbol{\alpha}}(P_A f)(\mathbf{0}) \right\|_B \\ &= \left\| \int_0^1 \sum_{j=1}^n D^{\boldsymbol{\alpha}+\boldsymbol{e}_j}(P_A f)(t\boldsymbol{z}_\alpha) z_{\boldsymbol{\alpha}}^{(j)}\, dt \right\|_B \\ &\leq \sum_{j=1}^n \|P_A f\|^B_{\mathbf{M},dA,S_{\boldsymbol{\gamma}}} (dA)^{|\boldsymbol{\alpha}|+1}(|\boldsymbol{\alpha}|+1)! M_{|\boldsymbol{\alpha}|+1} |z_{\boldsymbol{\alpha}}^{(j)}| \\ &\leq C|\mathcal{B} f|_{\mathbf{M},A,B} (dA)^{|\boldsymbol{\alpha}|+1}(|\boldsymbol{\alpha}|+1)! M_{|\boldsymbol{\alpha}|+1} \sum_{j=1}^n |z_{\boldsymbol{\alpha}}^{(j)}| \\ &\leq C\nu^{\circ}_{\mathbf{M},A}(\dot{f})(dA)^{|\boldsymbol{\alpha}|+1}(|\boldsymbol{\alpha}|+1)|\boldsymbol{\alpha}|! A_1^{|\boldsymbol{\alpha}|+1} M_1 M_{|\boldsymbol{\alpha}|} \sum_{j=1}^n |z_{\boldsymbol{\alpha}}^{(j)}|. \end{aligned}$$

According to (21), the previous amount is bounded by

$$\frac{k}{1+C}\nu^{\circ}_{\mathbf{M},A}(\dot{f}) A^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|},$$

so

$$|(\dot{\mathcal{D}} - \dot{\mathcal{B}})\dot{f}|_{\mathbf{M},A,B} = \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \frac{d_{\boldsymbol{\alpha}}}{A^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|}} \leq \frac{k}{1+C}\nu^{\circ}_{\mathbf{M},A}(\dot{f}),$$

and we have $\dot{\mathcal{D}}(\mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}}, B)) \subseteq \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ and

$$\dot{\mathcal{D}} : \mathcal{Q}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}}, B) \to \Lambda_{\mathbf{M},A}(\mathbb{N}_0^n, B)$$

is linear and continuous. In addition to this,

$$\|\dot{\mathcal{D}} - \dot{\mathcal{B}}\| \leq \frac{k}{1+C} < \frac{1}{1+C} \leq \frac{1}{\|\dot{\mathcal{B}}^{-1}\|},$$

so $\dot{\mathcal{D}}$ admits a continuous inverse. □

We can now give a rigidity result for the extension maps $T_{\mathbf{M},A,\boldsymbol{\gamma}}$.

**Theorem 4.2.** Under the assumptions in Lemma 4.1, if $\boldsymbol{a} \in \Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n, B)$ is such that

$$D^{\boldsymbol{\alpha}}(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}))(\boldsymbol{z}_{\boldsymbol{\alpha}}) = 0 \quad \text{for every } \boldsymbol{\alpha} \in \mathbb{N}_0^n,$$

then $\boldsymbol{a} \equiv 0$.

*Proof*:
The conditions on $\boldsymbol{a}$ allow us to write

$$\begin{aligned} \dot{\mathcal{D}}(\pi_{\mathbf{M},A} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})) &= (D^{\boldsymbol{\alpha}}(P_A(T_{\mathbf{M},A,\boldsymbol{\gamma}})(\boldsymbol{z}_{\boldsymbol{\alpha}})))_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \\ &= (D^{\boldsymbol{\alpha}}(T_{\mathbf{M},A,\boldsymbol{\gamma}} \circ \mathcal{B} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}})(\boldsymbol{z}_{\boldsymbol{\alpha}}))_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \\ &= (D^{\boldsymbol{\alpha}} T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{z}_{\boldsymbol{\alpha}}))_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} \equiv 0. \end{aligned}$$

By the previous lemma, $\dot{\mathcal{D}}$ admits inverse, so $\pi_{\mathbf{M},A} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}) = \dot{0}$, from where

$$\boldsymbol{a} = \mathcal{B} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a}) = \dot{\mathcal{B}}(\pi_{\mathbf{M},A} \circ T_{\mathbf{M},A,\boldsymbol{\gamma}}(\boldsymbol{a})) = \dot{\mathcal{B}}(\dot{0}) = 0,$$

as desired. $\square$

Combining the previous result with the ones on quasi-analyticity for the class $\mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}},B)$, we can state the following

**Corollary 4.3.** Let $n\in\mathbb{N}$, $\mathbf{M}$, $A>0$, $\boldsymbol{\gamma}\in(0,\infty)^n$ and $(\boldsymbol{z}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}$ be as in Lemma 4.1. Let $\boldsymbol{\theta}=(\theta_1,\dots,\theta_n)\in(0,\infty)^n$, put $\underline{\theta}=\min\{\theta_j:j\in\mathcal{N}\}$, and suppose that $\mathbf{M}$ verifies (7) and $\underline{\theta}\geq\gamma(\mathbf{M})$.

Let $f\in\mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}},B)$ be a map such that $\mathcal{B}f\in\Lambda^{\circ}_{\mathbf{M},A}(\mathbb{N}_0^n,B)$ and $D^{\boldsymbol{\alpha}}(T_{\mathbf{M},A,\boldsymbol{\gamma}}(\mathcal{B}f))(\boldsymbol{z}_{\boldsymbol{\alpha}})=0$ for every $\boldsymbol{\alpha}\in\mathbb{N}_0^n$. Then, $f$ is null in $S_{\boldsymbol{\theta}}$.

*Proof*:
It is clear that $\boldsymbol{a}:=\mathcal{B}f$ fulfills the hypotheses of the previous theorem, so we have $\boldsymbol{a}\equiv 0$. Now, whenever $\mathbf{M}$ verifies (7) and $\underline{\theta}\geq\gamma(\mathbf{M})$, we know from Proposition 2.14 that the class $\mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}},B)$ is quasi-analytic. This is enough to conclude. $\square$

We deal now with the rigidity of the map $U_{\mathbf{M},A,\boldsymbol{\gamma}}$ from Theorem 3.6, where $\mathbf{M}$, $A$ and $\boldsymbol{\gamma}$ are as before. Let us define $\mathfrak{F}^{\circ 1}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ as the set of families $\mathcal{G}=\{f_{jm}:j\in\mathcal{N},m\in\mathbb{N}_0\}\in\mathfrak{F}^{1}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$ (see Definition 3.3) such that

$$\lim_{m\to\infty}\frac{\|f_{jm}\|^{B}_{\mathbf{M},2AA_1,S_{\boldsymbol{\gamma}_{j'}}}}{(2AA_1)^m m! M_m}=0,\qquad \text{for every } j\in\mathcal{N}.$$

Equivalently, we can say that

$$\mathcal{G}=\{f_{jm}:j\in\mathcal{N},m\in\mathbb{N}_0\}\in\mathfrak{F}^{\circ 1}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$$

if, and only if, $\mathcal{G}$ is a coherent first order family and for every $j\in\mathcal{N}$ we have

$$\mathcal{G}_j:=(f_{jm})_{m\in\mathbb{N}_0}\in\Lambda^{\circ}_{\mathbf{M},2AA_1}(\mathbb{N}_0,\mathcal{A}_{\mathbf{M},2AA_1}(S_{\boldsymbol{\gamma}_{j'}},B)).$$

In the following result, the constants $c_j$ and $C_j$ $(j\in\mathcal{N})$ appearing are the ones obtained in the proof of Theorem 3.6.

**Theorem 4.4.** Under the previous assumptions, let $\mathcal{G}\in\mathfrak{F}^{\circ 1}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}},B)$, and denote by $H^{[1]},\dots,H^{[n]}$ the functions obtained after each step in the construction of $U_{\mathbf{M},A,\boldsymbol{\gamma}}(\mathcal{G})$, so that $H^{[n]}=U_{\mathbf{M},A,\boldsymbol{\gamma}}(\mathcal{G})$. Let us suppose there exists a family of complex numbers $\{z_{jm}:j\in\mathcal{N},m\in\mathbb{N}_0\}$ such that:

(i) For every $j\in\mathcal{N}$ we have $z_{jm}\in S_{\boldsymbol{\gamma}_j}$, for every $m\in\mathbb{N}_0$.

(ii) There exists $k\in(0,1)$ such that for every $j\in\mathcal{N}$ and every $m\in\mathbb{N}_0$,

$$C_j\Big(\prod_{\ell=1}^{j-1}c_\ell\Big)(2A_1)^j AM_1(m+1)(c_jA_1)^{m+1}|z_{jm}|\leq\frac{k}{1+C_j},$$

where $A_1$ is the constant in property $(\mu)$ for $\mathbf{M}$.

(iii) For every $j\in\mathcal{N}$ and $m\in\mathbb{N}_0$, $D^{m\boldsymbol{e}_j}(H^{[j]}(z_{jm},\cdot))$ is null in $S_{\boldsymbol{\gamma}_{j'}}$.

Then $\mathcal{G}$ is the null family.

*Proof*:
We will use the same notation as in Theorem 3.6. If $H_1^{[1]\star}=T_{\mathbf{M},2AA_1,\gamma_1}(\mathcal{G}_1)$ and $H^{[1]}(\boldsymbol{z})=H_1^{[1]\star}(z_1)(\boldsymbol{z}_{1'})$ for every $\boldsymbol{z}=(z_1)(\boldsymbol{z}_{1'})\in S_{\boldsymbol{\gamma}}$, taking into account (8) and condition (iii) we have $(H_1^{[1]\star})^{(m)}(z_{1m})=0$ for every $m\in\mathbb{N}_0$. Conditions (i) and (ii) allow us to apply Theorem 4.2 to $\mathcal{G}_1$, and so $\mathcal{G}_1$ is the null family. Due to $T_{\mathbf{M},2AA_1,\gamma_1}$ is a linear map we have $H_1^{[1]\star}=T_{\mathbf{M},2AA_1,\gamma_1}(\mathcal{G}_1)\equiv 0$, and then $H^{[1]}\equiv 0$ and $H_2^{[1]\star}\equiv 0$, so we also have $h_{2m}^{[1]}\equiv 0$ for every $m\in\mathbb{N}_0$. Then,

$$\mathcal{G}_2=(f_{2m}-h_{2m}^{[1]})_{m\in\mathbb{N}_0}=(f_{2m})_{m\in\mathbb{N}_0},$$

and $H_2^{[2]\star} = T_{\mathbf{M},c_1(2A_1)^2A,\gamma_2}(\mathcal{G}_2)$. The same argument may be repeated to obtain that $\mathcal{G}_2$ is the null family, and so on. □

We end this section with a combination of this result with Proposition 2.13 on (s) quasi-analyticity. Its proof is similar to that of Corollary 4.3, so we omit it.

**Corollary 4.5.** Let $n \in \mathbb{N}$, $\mathbf{M}$, $A > 0$ and $\boldsymbol{\gamma} \in (0,\infty)^n$ be as before. Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in (0,\infty)^n$ be such that $\gamma_j \le \theta_j$ for every $j \in \mathcal{N}$, let us put $\overline{\theta} = \max\{\theta_j : j \in \mathcal{N}\}$, and let us also suppose that $\mathbf{M}$ verifies (7) and $\overline{\theta} \ge \gamma(\mathbf{M})$.

Let $f \in \mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}}, B)$ be a map such that

(a) Its restriction to $S_{\boldsymbol{\gamma}}$, say $\tilde{f}$, is such that $\mathcal{B}_1(\tilde{f}) \in \mathfrak{F}^{\circ 1}_{\mathbf{M},A}(S_{\boldsymbol{\gamma}}, B)$.

(b) There exists a family of complex numbers $\{z_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$ that fulfills conditions (i), (ii) and (iii) from Theorem 4.4, where $H^{[1]}, H^{[2]}, \dots, H^{[n]}$ are the successive maps obtained in the construction of $U_{\mathbf{M},A,\boldsymbol{\gamma}}(\mathcal{B}_1(\tilde{f}))$ in Theorem 3.6, and $c_j, C_j$ are the constants there involved.

Then, $f$ is the null map in $S_{\boldsymbol{\theta}}$.

# 5 Proof of Lemma 3.5

This section is devoted to the sketch of the proof of Lemma 3.5.

**Notation**: For the sake of brevity, $B$ will stand for $\mathcal{A}_{\mathbf{M},A}(S_{\boldsymbol{\theta}})$. On the other hand, when necessary we identify the complex plane with $\mathbb{R}^2$ and the complex point $w = x + iy$ with the pair $(x, y)$.

We wish to point out that, as it was justified in V. Thilliez's work [18], under an appropriate ramification it is possible to reduce the construction of the extension operator $T_{\mathbf{M},A,\gamma}$ to the case that $\gamma < 2$. A study of the argument used there shows that we can also work under that assumption. The proof of this result is divided into different steps, and it follows the ones given by V. Thilliez in his construction. However, it was necessary for us to obtain accurate estimates in an exhaustive study of the construction of the operators $E_{A,B}$ and $F_{A,\Omega,B}$ given by J. Chaumat and A.-M. Chollet, which we made reference to in Proposition 2.7. These are the awkward calculations we will omit, limiting ourselves to state the precise information we can get at each moment:

(i) First of all, we associate a family $\boldsymbol{f}^{\mathbb{C}} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0^2, B)$ to a given $\boldsymbol{f} = (f_p)_{p\in\mathbb{N}_0} \in \Lambda_{\mathbf{M},A}(\mathbb{N}_0, B)$ by means of the equality

$$\sum_{(\ell,k)\in\mathbb{N}_0^2} f^{\mathbb{C}}_{(\ell,k)} \frac{x^\ell y^k}{\ell! k!} = \sum_{p\in\mathbb{N}_0} f_p \frac{(x+iy)^p}{p!},$$

and we construct $g^\star_{\boldsymbol{f}} := E_{A,B}(\boldsymbol{f}^{\mathbb{C}})$, with support contained in a disc $D = B(0, R_1)$. $g^\star_{\boldsymbol{f}}$ belongs to $\mathcal{C}_{\mathbf{M},c_1 A}(\mathbb{R}^2, B)$ (for an appropriate $c_1$) and is such that

$$D^{(p,0)} g^\star_{\boldsymbol{f}}(0,0) = f_p, \qquad p \in \mathbb{N}_0. \tag{22}$$

For every $m \in \mathbb{N}_0$ we obtain that

$$\lim_{z_j\to 0, z_j\in S_{\theta_j}} D^{m\boldsymbol{e}_j}\big(g^\star_{\boldsymbol{f}}(w)\big)(\boldsymbol{z}) = 0 \quad \text{uniformly for } w \in \mathbb{C},\ \boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}.$$

(ii) At the second stage we prove that, for every $m \in \mathbb{N}_0$ we prove that

$$\lim_{z_j\to 0, z_j\in S_{\theta_j}} D^{m\boldsymbol{e}_j}\big(\overline{\partial}_w g^\star_{\boldsymbol{f}}(w)\big)(\boldsymbol{z}) = 0 \quad \text{uniformly for } w \in \mathbb{C}, \boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}},$$

where the operator $\overline{\partial}_w$ is given by $\overline{\partial}_w = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$.

(iii) For suitably chosen $\tau > 0$ and $\gamma$, such that $0 < \gamma < \gamma(\mathbf{M})$, we define $\psi(w) = G(\tau w)$, $w \in S_\gamma$, where $G \in \mathcal{A}_{\mathbf{M}}(S_\gamma)$ is the complex function introduced in Theorem 2.3.1 and Lemma 2.3.2 in [18], whose derivatives and those of $1/G$ are suitably governed in terms of $\mathbf{M}$. We then have $\frac{1}{\psi}\overline{\partial}_w g^\star_{\boldsymbol{f}} \in \mathcal{C}_{\mathbf{M},c_2A}(S_\gamma, B)$ (for a certain $c_2$). At this stage, we get that

$$\lim_{z_j\to 0, z_j\in S_{\theta_j}} D^{m\boldsymbol{e}_j}\Big(\frac{1}{\psi(w)}\overline{\partial}_w g^\star_{\boldsymbol{f}}(w)\Big)(\boldsymbol{z}) = 0,$$

uniformly in $w \in S_\gamma$ and $\boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}$.

(iv) For $\Omega = D \cap S_\gamma$ we define the function

$$v^\star_{\boldsymbol{f}} = F_{c_3A,\Omega,B}(\frac{1}{\psi}\overline{\partial}_w g^\star_{\boldsymbol{f}}),$$

for an appropriate $c_3$, $F_{c_3A,\Omega,B}$ being the extension operator from Proposition 2.7, so that $v^\star_{\boldsymbol{f}} \in \mathcal{C}_{\mathbf{M},c_4A}(\mathbb{C}, B)$ for a certain $c_4$, it is equal to $\frac{1}{\psi}\overline{\partial}_w g^\star_{\boldsymbol{f}}$ in $\Omega$ and has its support contained in an open disc $D'$ centered at 0 and such that $\overline{D} \subseteq D'$. We prove that

$$\lim_{z_j\to 0, z_j\in S_{\theta_j}} D^{m\boldsymbol{e}_j}\big(v^\star_{\boldsymbol{f}}(w)\big)(\boldsymbol{z}) = 0$$

uniformly for $w \in \mathbb{C}$ and $\boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}$.

(v) Let us consider a map $\chi \in \mathcal{C}_{\mathbf{M},A}(\mathbb{C})$ with support contained in $D'$ and identically equal to 1 in $D$. We then have

$$\chi v^\star_{\boldsymbol{f}} = \frac{1}{\psi}\overline{\partial}_w g^\star_{\boldsymbol{f}} \quad \text{in } S_\gamma. \tag{23}$$

Let $\mathcal{K}(w) = -1/(\pi w)$, $w \in \mathbb{C}\setminus\{0\}$; we define $u^\star_{\boldsymbol{f}} = \mathcal{K} * (\chi v^\star_{\boldsymbol{f}})$ (where $*$ denotes convolution), which satisfies

$$\overline{\partial}_w u^\star_{\boldsymbol{f}} = \chi v^\star_{\boldsymbol{f}} \quad \text{in } \mathbb{C}. \tag{24}$$

We obtain that

$$\lim_{z_j\to 0, z_j\in S_{\theta_j}} D^{m\boldsymbol{e}_j}\Big(u^\star_{\boldsymbol{f}}(w)\Big)(\boldsymbol{z}) = 0,$$

uniformly for $w \in \mathbb{C}$ and $\boldsymbol{z}_{j'} \in S_{\boldsymbol{\theta}_{j'}}$.

(vi) For an adequate $c_5$ we have that $\psi u^\star_{\boldsymbol{f}}$ belongs to $\mathcal{C}_{\mathbf{M},c_5A}(S_\gamma, B)$ and, moreover, it is flat at the origin, since $\psi$ is. We define $T_{\mathbf{M},A,\gamma}\boldsymbol{f} := g^\star_{\boldsymbol{f}} - \psi u^\star_{\boldsymbol{f}}$. According to (22), its derivatives at the origin are the desired ones, and it is holomorphic in $S_\gamma$ by (23) and (24). Finally, we deduce (18).

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