Linear continuous extension operators
for Gevrey classes on polysectors

J. Sanz

Abstract: We study Gevrey classes of holomorphic functions of several variables on a polysector, and their relation to classes of Gevrey strongly asymptotically developable functions. A new Borel-Ritt-Gevrey interpolation problem is formulated, and its solution is obtained by the construction of adequate linear continuous extension operators. Our results improve those given by Haraoka in this context, and extend to several variables the one-dimensional versions of the Borel-Ritt-Gevrey theorem given by Ramis and Thilliez, respectively. Some rigidity properties for the constructed operators are stated.

AMS Mathematics Subject Classification: 30D60, 30E05, 41A60.

1 Introduction

A holomorphic complex function $f = f(z)$ on a sector S in the complex plane with vertex at 0 admits s-Gevrey asymptotic expansion, given by the (formal) series $\sum_{m=0}^{\infty} a_m z^m$, as z tends to 0 if, and only if, the derivatives of f are subject to s-Gevrey type bounds on proper subsectors of S. In this case, $a_m = \lim_{z\to 0} \frac{1}{m!} f^{m)}(z)$, so that $\sum_{m=0}^{\infty} a_m z^m$ is an s-Gevrey series. Conversely, for sectors S_θ of suitably small opening θ, the Borel-Ritt-Gevrey theorem (see [11], [8], [1, 2.2.1]) guarantees the existence of s-Gevrey holomorphic functions on S_θ having an arbitrarily prescribed s-Gevrey asymptotic expansion. Thilliez [13] has obtained a similar result, which may be seen as a linear continuous version of the Borel-Ritt-Gevrey theorem, by constructing an extension operator from the space of Gevrey series into the space of functions whose derivatives admit Gevrey-like bounds uniformly on all of S_θ (see Section 3).

Regarding functions of several complex variables, the concept of strong asymptotic developability given by Majima [6, 7] resembles the one-variable definition in the sense that, for a function f holomorphic on a polysector $S \subset \mathbb{C}^n$ with vertex at $\mathbf{0}$, to be strongly asymptotically developable amounts to the boundedness of the derivatives of f on proper subpolysectors of S (cf. [5, 12]). The asymptotic behaviour of f is determined by the family $TA(f)$, consisting of functions obtained as limits of the derivatives of f when some of its variables tend to 0. Haraoka [3] studied this concept for Gevrey functions of several variables, and gave two interpolation results starting from Gevrey data of the same type as

$$FA(f) = \{f_{\boldsymbol{\alpha}} = \lim_{\boldsymbol{z}\to 0} \frac{1}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\}_{\boldsymbol{\alpha}\in\mathbb{N}^n} \subset TA(f).$$

However, in general, for a function f strongly asymptotically developable the knowledge of $FA(f)$ does not allow to recover $TA(f)$; so, one should put an interpolation problem taking as initial datum a family $\mathcal{F}$ of the same type as $TA(f)$ and subject to natural conditions that assure there might be a function f with $TA(f) = \mathcal{F}$. As far as we know, no such problem has been studied in the case of Gevrey functions.

The main purpose of this paper is to give the solution for a new interpolation problem of the said nature, by means of the construction of linear continuous extension operators. Theorem 3.4 extends the result of Thilliez for one variable functions [13, Theorem 1.3] to the several variables case. Though our statements remain valid for polysectors in $\mathcal{R}^n$, where $\mathcal{R}$ is the Riemann surface of $\log(z)$, we restrict our attention to polysectors in $\mathbb{C}^n$.

After giving some notation (Section 2), the problem and its solution are stated in Section 3. For $\boldsymbol{\sigma} \in (0,\infty)^n$, $\boldsymbol{s} \in (1,\infty)^n$, a polysector $S_{\boldsymbol{\theta}} \subset \mathbb{C}^n$ of opening $\boldsymbol{\theta} \in (0, 2\pi)^n$ and a Banach space E, we consider the space $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ of holomorphic functions $f\colon S_{\boldsymbol{\theta}} \to E$ such that

$$\|f\|_{\boldsymbol{\sigma}} := \sup_{\boldsymbol{z}\in S_{\boldsymbol{\theta}},\ \boldsymbol{\alpha}\in\mathbb{N}^n} \frac{\|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\|}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}} \boldsymbol{\sigma}^{\boldsymbol{\alpha}}} < +\infty;$$

its relation to spaces of functions $\boldsymbol{s}$-Gevrey strongly asymptotically developable is studied. For $f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, we denote by $TA^{\diamond}(f)$ the family consisting of those elements of $TA(f)$ in $n-1$ variables. $TA^{\diamond}(f)$ uniquely determines $TA(f)$, satisfies certain coherence conditions, and their elements are subject to special Gevrey-type bounds. This leads us to define the appropriate data space $G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, so arriving at the main result in this paper, Theorem 3.4:

If $\boldsymbol{\theta} < (\boldsymbol{s} - \mathbf{1})\pi$, then there exist $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}) = (c_1(s_1, \theta_1), \ldots, c_n(s_n, \theta_n)) \in (1, \infty)^n$, $C = C(\boldsymbol{s}, \boldsymbol{\theta}) > 0$ and, for each $\boldsymbol{\sigma}$, a linear operator

$$U_{\boldsymbol{\sigma},\boldsymbol{\theta}}\colon G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\theta}}, E)$$

such that for every $\mathcal{G} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$,

$$TA^{\diamond}(U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})) = \mathcal{G} \quad \text{and} \quad \|U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})\|_{\boldsymbol{c\sigma}} \leq C N_{\boldsymbol{\sigma}}(\mathcal{G}),$$

where $\|\cdot\|_{\boldsymbol{c\sigma}}$ and $N_{\boldsymbol{\sigma}}$ are the respective norms in the considered spaces.

Section 4 is mainly devoted to the proof of this theorem. We first obtain a result (Theorem 4.1), similar to that of Thilliez [13, Theorem 1.3], for vector valued functions of one variable. The technique, elementary and completely different from that of Thilliez, is based on Ramis' one [11, 8], which allows a suitable study of the bounds; it is also amenable to the determination of the behaviour of the interpolating function in the special case when the space E in which it takes its values is of the type $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ (Lemma 4.3). Now, the fact that the spaces $\mathcal{W}^{(\boldsymbol{s},\boldsymbol{t})}_{(\boldsymbol{\sigma},\boldsymbol{\tau})}(S_{\boldsymbol{\theta}} \times U_{\boldsymbol{\varphi}}, E)$ and $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, \mathcal{W}^{\boldsymbol{t}}_{\boldsymbol{\tau}}(U_{\boldsymbol{\varphi}}, E))$ are isomorphic (Proposition 4.2), combined with a repeated application of Theorem 4.1 and Lemma 4.3, lets us apply a recurrent argument on the number of variables to obtain Theorem 3.4. As another consequence of Proposition 4.2, we give, in this context, a linear continuous version (Theorem 3.3) of the first interpolation result proven by Haraoka ([3, Theorem 1.(1)]; see Section 3).

We emphasize that the consideration of vector valued functions in this paper makes no difference as to the difficulty of the proofs; it is only due to the need for the isomorphism just mentioned.

Finally, some rigidity properties for the different extension operators are stated in Section 5. Annihilation conditions are given on the extending functions that assure the initial data are null. We adopt the setting of the problem from the work of Thilliez [13], and obtain similar results for functions of several variables. While Thilliez's method rests on a theorem of Paley-Wiener type dealing with Schauder bases, this does not seem to apply here, and the main result in this Section, Theorem 5.2, is based on the direct proof that a certain operator is invertible (Lemma 5.1).

2 Notation

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $n \geq 1$, put $N = \{1, 2, \ldots, n\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$ be two multiindices, $m \in [0, \infty)$, $\boldsymbol{t} = (t_1, t_2, \ldots, t_n) \in [0, \infty)^n$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. We set

$$\begin{array}{ll}
\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n) & m\boldsymbol{t} = (mt_1, mt_2, \ldots, mt_n) \\
|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n & \boldsymbol{\alpha}! = \alpha_1!\,\alpha_2! \cdots \alpha_n! \\
\boldsymbol{\alpha} \leq \boldsymbol{\beta} \Leftrightarrow \alpha_j \leq \beta_j,\ j \in N & \boldsymbol{\alpha} < \boldsymbol{\beta} \Leftrightarrow \alpha_j < \beta_j,\ j \in N \\
\mathbf{1} = (1, 1, \ldots, 1) & \boldsymbol{e}_j = (0, \ldots, \overset{j)}{1}, \ldots, 0) \\
|\boldsymbol{z}^{\boldsymbol{\alpha}}| = |\boldsymbol{z}|^{\boldsymbol{\alpha}} = |z_1|^{\alpha_1} |z_2|^{\alpha_2} \cdots |z_n|^{\alpha_n} & \boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\
D^{\boldsymbol{\alpha}} = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}} &
\end{array}$$

If J is a nonempty subset of N, the number of elements of J will be $\#J$.

Consider, for $j = 1, 2, \ldots, n$, an open sector in $\mathbb{C}$ with vertex at the origin,

$$S_j = \{\, z \in \mathbb{C}\colon \theta_{1j} < \arg(z) < \theta_{2j} \,\}, \quad 0 < \theta_{2j} - \theta_{1j} < 2\pi.$$

Any cartesian product $S = \prod_{j=1}^{n} S_j \subset \mathbb{C}^n$ of open sectors in $\mathbb{C}$ with vertex at 0 will be called an (unbounded open) *polysector* in $\mathbb{C}^n$ with vertex at $\mathbf{0}$.

We say a polysector T in $\mathbb{C}^n$ (with vertex at the origin) is a *proper subpolysector* of S if $T = \prod_{j=1}^{n} T_j$ with $\overline{T_j} \subset S_j \cup \{0\}$, $j = 1, 2, \ldots, n$. For convenience, polysectors of the form

$$S_{\boldsymbol{\theta}} = \{\, \boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n\colon |\mathrm{Arg}(z_j)| < \frac{\theta_j}{2},\ j = 1, \ldots, n \,\}$$

will be mostly considered. We say $\boldsymbol{\theta} \in (0,2\pi)^n$ is the *opening* of $S_{\boldsymbol{\theta}}$.

If J is a nonempty subset of N and $\boldsymbol{z} \in \mathbb{C}^n$, we write $\boldsymbol{z}_J$ for the restriction of $\boldsymbol{z}$ to J, regarding $\boldsymbol{z}$ as an element of $\mathbb{C}^N$. Let J and L be nonempty disjoint subsets of N. For $\boldsymbol{z}_J \in \mathbb{C}^J$ and $\boldsymbol{z}_L \in \mathbb{C}^L$, $(\boldsymbol{z}_J,\boldsymbol{z}_L)$ represents the element of $\mathbb{C}^{J\cup L}$ satisfying $(\boldsymbol{z}_J,\boldsymbol{z}_L)_J=\boldsymbol{z}_J$, $(\boldsymbol{z}_J,\boldsymbol{z}_L)_L=\boldsymbol{z}_L$; we also write $J'=N-J$, and for $j\in N$ we use j' instead of $\{j\}'$. In particular, we shall use these conventions for multiindices.

Finally, if $S_{\boldsymbol{\theta}}=\prod_{j=1}^n S_{\theta_j}$ is a polysector of $\mathbb{C}^n$, then $S_{\boldsymbol{\theta}_J}=\prod_{j\in J} S_{\theta_j}\subset\mathbb{C}^J$.

3 Preliminaries and results on extension operators

Let $(E,\|\cdot\|)$ be a complex Banach space, $\boldsymbol{\theta}\in(0,2\pi)^n$, $\boldsymbol{s}\in(1,\infty)^n$ and $\boldsymbol{\sigma}\in(0,\infty)^n$. Denote by $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$ the complex vector space consisting of the holomorphic functions $f\colon S_{\boldsymbol{\theta}}\to E$ such that

$$\|f\|_{\boldsymbol{\sigma}} := \sup_{\boldsymbol{z}\in S_{\boldsymbol{\theta}},\ \boldsymbol{\alpha}\in\mathbb{N}^n} \frac{\|D^{\boldsymbol{\alpha}}f(\boldsymbol{z})\|}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}} < +\infty.$$

$(\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E),\|\cdot\|_{\boldsymbol{\sigma}})$ is a Banach space. For instance, we note that, in the simpler case $E=\mathbb{C}$, non-constant polynomical functions do not belong to any $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},\mathbb{C})$, while functions such as ψ, given by

$$\psi(\boldsymbol{z})=\prod_{j=1}^n \exp(-z_j^{-1/(s_j-1)}),$$

belong to $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},\mathbb{C})$, for suitable $\boldsymbol{\sigma}$, whenever $\boldsymbol{\theta}<(\boldsymbol{s}-\boldsymbol{1})\pi$.

Let $f\in\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$; since all its derivatives are bounded on $S_{\boldsymbol{\theta}}$, Barrow's formula implies that they are lipschitzian. Hence, for $\emptyset\neq J\subsetneq N$ and $\boldsymbol{\alpha}_J\in\mathbb{N}^J$ (respectively, for $J=N$ and $\boldsymbol{\alpha}_N=\boldsymbol{\alpha}\in\mathbb{N}^N\equiv\mathbb{N}^n$) we can define a function from $S_{\boldsymbol{\theta}_{J'}}$ to E (resp. a constant in E) by

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J'})=\lim_{\substack{\boldsymbol{z}_J\to\boldsymbol{0}\\ \boldsymbol{z}_J\in S_{\boldsymbol{\theta}_J}}}\frac{D^{(\boldsymbol{\alpha}_J,\boldsymbol{0}_{J'})}f(\boldsymbol{z})}{\boldsymbol{\alpha}_J!}\quad(\text{resp. } f_{\boldsymbol{\alpha}}=\lim_{\substack{\boldsymbol{z}\to\boldsymbol{0}\\ \boldsymbol{z}\in S_{\boldsymbol{\theta}}}}\frac{D^{\boldsymbol{\alpha}}f(\boldsymbol{z})}{\boldsymbol{\alpha}!}); \tag{1}$$

the limit is uniform on $S_{\boldsymbol{\theta}_{J'}}$ whenever $J\neq N$, what implies that $f_{\boldsymbol{\alpha}_J}\in\mathcal{W}^{\boldsymbol{s}_{J'}}_{\boldsymbol{\sigma}_{J'}}(S_{\boldsymbol{\theta}_{J'}},E)$. If we adopt the convention that $\mathcal{W}^{\boldsymbol{s}_{N'}}_{\boldsymbol{\sigma}_{N'}}(S_{\boldsymbol{\theta}_{N'}},E)=E$, we also deduce that for every $\emptyset\neq J\subset N$ and $\boldsymbol{\alpha}_J\in\mathbb{N}^J$ we have

$$\|f_{\boldsymbol{\alpha}_J}\|_{\boldsymbol{\sigma}_{J'}}\leq(\boldsymbol{\alpha}_J!)^{\boldsymbol{s}_J-\boldsymbol{1}}\boldsymbol{\sigma}_J^{\boldsymbol{\alpha}_J}\|f\|_{\boldsymbol{\sigma}}. \tag{2}$$

In this way we may associate with f a family

$$TA_0(f)=\{\,f_{\boldsymbol{\alpha}_J}\colon\emptyset\neq J\subset N,\ \boldsymbol{\alpha}_J\in\mathbb{N}^J\,\},$$

which we call the *derived family* for f. The limits in (1) being uniform, we obtain

Proposition 3.1 (Coherence conditions) *Let $f\in\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$. Then, for every pair of disjoint nonempty subsets J and L of N, $\boldsymbol{\alpha}_J\in\mathbb{N}^J$ and $\boldsymbol{\alpha}_L\in\mathbb{N}^L$,*

$$\lim_{\substack{\boldsymbol{z}_L\to\boldsymbol{0}\\ \boldsymbol{z}_L\in S_{\boldsymbol{\theta}_L}}}\frac{D^{(\boldsymbol{\alpha}_L,\boldsymbol{0}_{(J\cup L)'})}f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J'})}{\boldsymbol{\alpha}_L!}=f_{(\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)'}); \tag{3}$$

the limit is uniform on $S_{\boldsymbol{\theta}_{(J\cup L)'}}$ whenever $J\cup L\neq N$.

Hereafter, we will say that a family

$$\mathcal{F}=\{\,f_{\boldsymbol{\alpha}_J}\in\mathcal{W}^{\boldsymbol{s}_{J'}}_{\boldsymbol{\sigma}_{J'}}(S_{\boldsymbol{\theta}_{J'}},E)\colon\emptyset\neq J\subset N,\ \boldsymbol{\alpha}_J\in\mathbb{N}^J\,\},$$

or briefly $\mathcal{F}=\{\,f_{\boldsymbol{\alpha}_J}\,\}$, is *coherent* if it verifies (3).

Denote by $\mathcal{A}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$ the complex vector space of the holomorphic functions $f\colon S_{\boldsymbol{\theta}}\to E$ such that there exists a family

$$TA(f)=\left\{\,f_{\boldsymbol{\alpha}_J}\colon\emptyset\neq J\subset N,\ \boldsymbol{\alpha}_J\in\mathbb{N}^J\,\right\},$$

where $f_{\boldsymbol{\alpha}_J}$ is a holomorphic function from $S_{\boldsymbol{\theta}_{J'}}$ to E when $J \neq N$, and $f_{\boldsymbol{\alpha}_J} \in E$ when $J = N$, satisfying the following: if we define

$$App_{\boldsymbol{\alpha}}(TA(f))(\boldsymbol{z}) = \sum_{\emptyset \neq J \subset N} (-1)^{\#J+1} \sum_{\substack{\boldsymbol{\beta}_J \in \mathbb{N}^J \\ \boldsymbol{\beta}_J \leq \boldsymbol{\alpha}_J - \mathbf{1}_J}} f_{\boldsymbol{\beta}_J}(\boldsymbol{z}_{J'}) \boldsymbol{z}_J^{\boldsymbol{\beta}_J}, \quad \boldsymbol{\alpha} \in \mathbb{N}^n, \ \boldsymbol{z} \in S_{\boldsymbol{\theta}},$$

then

$$M_{\boldsymbol{\sigma}}(f) := \sup_{\boldsymbol{z} \in S_{\boldsymbol{\theta}}, \ \boldsymbol{\alpha} \in \mathbb{N}^n} \frac{\|f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(TA(f))(\boldsymbol{z})\|}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}-\mathbf{1}} \boldsymbol{\sigma}^{\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}} < +\infty. \tag{4}$$

$TA(f)$ turns out to be unique, and will be called the *total family of strongly asymptotic expansion* associated to f. The subfamily $\{ f_{\boldsymbol{\alpha}_N} = f_{\boldsymbol{\alpha}} \in E \colon \boldsymbol{\alpha} \in \mathbb{N}^n \} \subset TA(f)$ will be denoted by $FA(f)$.

We note that when $n = 1$, given $s > 1$ and $\theta \in (0, 2\pi)$, we have $f \in \mathcal{A}^s_\sigma(S_\theta, E)$ if and only if there exists a family $TA(f) = FA(f) = \{ a_m \in E \colon m \in \mathbb{N} \}$ (or, equivalently, a formal power series $\sum_{m=0}^\infty a_m z^m$) such that

$$\sup_{z \in S_\theta, \ m \in \mathbb{N}} \frac{\|f(z) - \sum_{p=0}^{m-1} a_p z^p\|}{(m!)^{s-1} \sigma^m |z|^m} < +\infty.$$

In this situation we write $f \sim \sum_{m=0}^\infty a_m z^m$.

Some remarks are in order. The concept of strong asymptotic developability was established by Majima [6], and Haraoka [3] adapted it to the case of Gevrey functions. In the present context, Haraoka's definition would read as follows: a holomorphic function $f \colon S_{\boldsymbol{\theta}} \to E$ is $\boldsymbol{s}$-*Gevrey strongly asymptotically developable* as $\boldsymbol{z}$ tends to 0 in $S_{\boldsymbol{\theta}}$ (we write $f \in \mathcal{A}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$) if the suprema in (4) above, when taken over each proper subpolysector T of $S_{\boldsymbol{\theta}}$ and for a suitable $\boldsymbol{\sigma} = \boldsymbol{\sigma}(T)$, are finite (and depend on T).

With a similar change in the definition of the space $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we obtain a new vector space, $\mathcal{W}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$. Haraoka [3, §1, Proposition 3] proved that the following statements (with $E = \mathbb{C}$, but this makes no difference) are equivalent:

(i) $f \in \mathcal{A}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$;

(ii) $f \in \mathcal{W}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$ and f is strongly asymptotically developable as $\boldsymbol{z}$ tends to 0 in $S_{\boldsymbol{\theta}}$ (in the sense of Majima).

Indeed, the asumption in (ii) that f be strongly asymptotically developable is removable, as it can be easily deduced from Theorem 3.2 in [12]. In our situation we immediately have

Proposition 3.2 a) *$\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \subset \mathcal{A}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, and for every $f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we have $TA(f) = TA_0(f)$ and $M_{\boldsymbol{\sigma}}(f) \leq \|f\|_{\boldsymbol{\sigma}}$. Also, for every $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}^{\boldsymbol{s}_{J'}}_{\boldsymbol{\sigma}_{J'}}(S_{\boldsymbol{\theta}_{J'}}, E)$, and by Proposition 3.1,*

$$TA(f_{\boldsymbol{\alpha}_J}) = TA_0(f_{\boldsymbol{\alpha}_J}) = \{ f_{(\boldsymbol{\alpha}_J, \boldsymbol{\beta}_L)} \colon \emptyset \neq L \subset J', \ \boldsymbol{\beta}_L \subset \mathbb{N}^L \}.$$

b) *For every $\boldsymbol{\varphi} \in (0, \infty)^n$ with $\boldsymbol{\varphi} < \boldsymbol{\theta}$, denote by $R_{\boldsymbol{\varphi}} f$ the restriction to $S_{\boldsymbol{\varphi}}$ of functions f defined on $S_{\boldsymbol{\theta}}$. Then, there exists $\boldsymbol{c} = (c_1, \ldots, c_n) \in (1, \infty)^n$, where $c_j = c_j(\theta_j, \varphi_j)$, such that $R_{\boldsymbol{\varphi}}(\mathcal{A}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)) \subset \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\varphi}}, E)$, and for every $f \in \mathcal{A}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, $\|R_{\boldsymbol{\varphi}} f\|_{\boldsymbol{c\sigma}} \leq M_{\boldsymbol{\sigma}}(f)$. Here, $\boldsymbol{c\sigma}$ means $(c_1\sigma_1, \ldots, c_n\sigma_n)$.*

For $n \geq 1$, let us define

$$\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E) = \{ \underline{a} = \{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathbb{N}^n} \colon \nu_{\boldsymbol{\sigma}}(\underline{a}) := \sup_{\boldsymbol{\alpha} \in \mathbb{N}^n} \frac{\|a_{\boldsymbol{\alpha}}\|}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}-\mathbf{1}} \boldsymbol{\sigma}^{\boldsymbol{\alpha}}} < +\infty \};$$

$(\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E), \nu_{\boldsymbol{\sigma}})$ is a Banach space. Note that whenever $f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we have $FA(f) \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$ (we trivially identify families and multisequences), and the map $\mathcal{J} \colon \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$ sending f to $FA(f)$ is linear and, by (2), continuous, with $\|\mathcal{J}\| \leq 1$. In case $f \in \mathcal{A}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$ we have $FA(f) \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$ for some $\boldsymbol{\sigma} \in (0, \infty)^n$.

The following interpolation problem arises: given $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$, find a holomorphic function $f \colon S_{\boldsymbol{\theta}} \to E$ that belongs to some of the Gevrey spaces considered and such that $FA(f) = \underline{a}$.

For $n = 1$, Ramis (cf. [10, 11, 8]; see also [1, Proposition 2.2.1]) showed that, whenever $\theta \leq (s-1)\pi$, the problem is solvable in $\mathcal{A}^s(S_\theta, \mathbb{C})$. Haraoka generalized this result to higher dimensions [3, §2, Theorem 1.(1)]:

If $\boldsymbol{\theta} < (\boldsymbol{s} - \mathbf{1})\pi$, then for every $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(\mathbb{C})$, there exists $f \in \mathcal{A}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, \mathbb{C})$ such that $FA(f) = \underline{a}$. Their methods rely on the use of the one- or multi-dimensional finite Laplace transform; no explicit information is given on the relation between $\boldsymbol{\sigma}$ and the constants $\boldsymbol{\sigma}_T$ in (4) corresponding to each proper sub(poly)sector T of $S_{\boldsymbol{\theta}}$.

Also, for $n=1$, Thilliez [13, Theorem 1.3] proves that if $\theta \in (0,2\pi)$ and $\theta < (s-1)\pi$, then for every $\sigma > 0$ there exist constants $c = c(\theta, s) \geq 1$ and $C = C(\theta, s, \sigma) > 0$, and a linear operator $T_{\sigma,\theta}\colon \Gamma^s_\sigma(\mathbb{C}) \to \mathcal{W}^s_{c\sigma}(S_\theta, \mathbb{C})$ such that for every $\underline{a} \in \Gamma^s_\sigma(\mathbb{C})$, $\mathcal{J}T_{\sigma,\theta}(\underline{a}) = \underline{a}$ and $\|T_{\sigma,\theta}(\underline{a})\|_{c\sigma} \leq C\nu_\sigma(\underline{a})$. The reason why $\theta < (s-1)\pi$ in this result is that bounds for the derivatives of the solution need to be uniform on all of S_θ. The technique now rests on results on continuous extensions in ultradifferentiable classes of functions (see [2]).

We will obtain the following generalization for functions of several variables.

Theorem 3.3 *Let $\boldsymbol{s} \in (1,\infty)^n$ and $\boldsymbol{\theta} \in (0,2\pi)^n$ with $\boldsymbol{0} < \boldsymbol{\theta} < (\boldsymbol{s}-\boldsymbol{1})\pi$. Then, there exist $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{s},\boldsymbol{\theta}) = (c_1(s_1,\theta_1),\dots,c_n(s_n,\theta_n)) \in (1,\infty)^n$, $C = C(\boldsymbol{s},\boldsymbol{\theta}) > 0$ and, for each $\boldsymbol{\sigma} \in (0,\infty)^n$, a linear map $T_{\boldsymbol{\sigma},\boldsymbol{\theta}}\colon \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E) \to \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\theta}}, E)$ such that for every $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$,*

$$\mathcal{J}(T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})) = \underline{a}, \qquad \|T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})\|_{\boldsymbol{c\sigma}} \leq C\,\nu_{\boldsymbol{\sigma}}(\underline{a}).$$

As said before, regarding functions of several variables, one should consider the following question: is it possible to interpolate starting from the whole family $TA(f)$? As far as we know, the only result in this sense was proven by Haraoka [3, §2, Theorem 1.(2)]: under the additional hypothesis that $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$ verifies some convergence conditions which make possible to obtain a whole family $\mathcal{F} = \{f_{\boldsymbol{\alpha}_J}\}$ by means of the relations

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J'}) = \sum_{\boldsymbol{\beta}_{J'} \in \mathbb{N}^{J'}} a_{(\boldsymbol{\alpha}_J,\boldsymbol{\beta}_{J'})} \boldsymbol{z}_{J'}^{\boldsymbol{\beta}_{J'}},$$

the existence is proven of a function $f \in \mathcal{A}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, \mathbb{C})$ such that $TA(f) = \mathcal{F}$.

We now give the framework needed to answer the previous question in the affirmative. For $n > 1$ and $f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we call

$$TA^\diamond(f) = \{\, f_{m_{\{j\}}} \in \mathcal{W}^{\boldsymbol{s}_{j'}}_{\boldsymbol{\sigma}_{j'}}(S_{\boldsymbol{\theta}_{j'}}, E) \colon j \in N,\ m \in \mathbb{N} \,\}$$

the *first order family* associated to f. It consists of those elements of $TA(f)$ in $n-1$ variables. For convenience, we write f_{jm} instead of $f_{m_{\{j\}}}$. $TA(f)$ is coherent, hence $TA^\diamond(f)$ satisfies the following *first order coherence conditions*:

for every $L \subset N$ consisting of at least two elements, every $\boldsymbol{\alpha}_L \in \mathbb{N}^L$ and every $j, \ell \in L$, we have

$$\lim_{\substack{\boldsymbol{z}_{L-\{j\}} \to 0 \\ \boldsymbol{z}_{L-\{j\}} \in S_{\boldsymbol{\theta}_{L-\{j\}}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{j\}},\boldsymbol{0}_{L'})} f_{j\alpha_j}(\boldsymbol{z}_{j'})}{\boldsymbol{\alpha}_{L-\{j\}}!} = \lim_{\substack{\boldsymbol{z}_{L-\{\ell\}} \to 0 \\ \boldsymbol{z}_{L-\{\ell\}} \in S_{\boldsymbol{\theta}_{L-\{\ell\}}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{\ell\}},\boldsymbol{0}_{L'})} f_{\ell\alpha_\ell}(\boldsymbol{z}_{\ell'})}{\boldsymbol{\alpha}_{L-\{\ell\}}!};$$

the limits are uniform on $S_{\boldsymbol{\theta}_{L'}}$ whenever $L \neq N$.

$TA^\diamond(f)$ determines $TA(f)$ uniquely. Conversely, if we consider a family

$$\mathcal{F}^\diamond = \{\, f_{jm} \in \mathcal{W}^{\boldsymbol{s}_{j'}}_{\boldsymbol{\sigma}_{j'}}(S_{\boldsymbol{\theta}_{j'}}, E) \colon j \in N,\ m \in \mathbb{N} \,\}$$

under the first order coherence conditions (we will say $\mathcal{F}^\diamond = \{\, f_{jm} \,\}$ is a *coherent first order family*), we may construct in a unique way a coherent family $\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \,\}$ whose first order subfamily is $\mathcal{F}^\diamond$ (for details, see [12]).

Moreover, for $f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we have from (2) that

$$\{f_{jm}\}_{m=0}^\infty \in \Gamma^{s_j}_{\sigma_j}\big(\mathcal{W}^{\boldsymbol{s}_{j'}}_{\boldsymbol{\sigma}_{j'}}(S_{\boldsymbol{\theta}_{j'}}, E)\big), \quad \nu_{\sigma_j}(\{f_{jm}\}_{m=0}^\infty) \leq \|f\|_{\boldsymbol{\sigma}}, \qquad j \in N.$$

So, we are led to define the space $G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ consisting of the coherent first order families $\mathcal{G} = \{f_{jm}\}$ such that for every $j \in N$, $\mathcal{G}_j = \{f_{jm}\}_{m=0}^\infty \in \Gamma^{s_j}_{\sigma_j}\big(\mathcal{W}^{\boldsymbol{s}_{j'}}_{\boldsymbol{\sigma}_{j'}}(S_{\boldsymbol{\theta}_{j'}}, E)\big)$. Setting $N_{\boldsymbol{\sigma}}(\mathcal{G}) = \sup_{j\in N} \nu_{\sigma_j}(\mathcal{G}_j)$, $(G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E), N_{\boldsymbol{\sigma}})$ is a Banach space, and the map $\mathcal{J}_1\colon \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ sending f to $TA^\diamond(f)$ is linear, continuous and $\|\mathcal{J}_1\| \leq 1$. As a converse, we will obtain the following linear continuous version of the generalized Borel-Ritt-Gevrey theorem for functions of several variables.

Theorem 3.4 *Given $\boldsymbol{s} \in (1,\infty)^n$ and $\boldsymbol{\theta} \in (0,2\pi)^n$ with $\boldsymbol{\theta} < (\boldsymbol{s}-\boldsymbol{1})\pi$, there exist a constant vector $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{s},\boldsymbol{\theta}) = (c_1(s_1,\theta_1),\dots,c_n(s_n,\theta_n)) \in (1,\infty)^n$, a constant $C = C(\boldsymbol{s},\boldsymbol{\theta}) > 0$ and, for each $\boldsymbol{\sigma} \in (0,\infty)^n$, a linear operator*

$$U_{\boldsymbol{\sigma},\boldsymbol{\theta}}\colon G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \longrightarrow \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\theta}}, E)$$

such that for every $\mathcal{G} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ we have

$$\mathcal{J}_1(U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})) = \mathcal{G} \qquad \text{and} \qquad \|U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})\|_{\boldsymbol{c\sigma}} \leq C N_{\boldsymbol{\sigma}}(\mathcal{G}).$$

4 Proofs for results in Section 3

We recall that in the one-dimensional case the families $TA(f)$ and $FA(f)$ coincide.

Theorem 4.1 *Let $s\in\mathbb{R}, s>1$; $S=\{z\in\mathbb{C}:|\mathrm{Arg}(z)|<\min(\pi,(s-1)\frac{\pi}{2})\}$, and E a complex Banach space. Then, for every $\sigma>0$ there exists a linear operator $T_\sigma:\Gamma^s_\sigma(E)\to\mathcal{A}^s(S,E)$ such that for every $\underline{a}\in\Gamma^s_\sigma(E)$ we have $TA(T_\sigma(\underline{a}))=FA(T_\sigma(\underline{a}))=\underline{a}$; for every $\theta\in(0,2\pi)$ with $\theta<(s-1)\pi$ there exist $c=c(s,\theta)>1$ and $C=C(s,\theta)>0$ such that for every $\sigma>0$, $T_{\sigma,\theta}=R_\theta\circ T_\sigma$ (R_θ denotes restriction to S_θ) maps $\Gamma^s_\sigma(E)$ into $\mathcal{W}^s_{c\sigma}(S_\theta,E)$, $\mathcal{J}\circ T_{\sigma,\theta}$ is the identity map on $\Gamma^s_\sigma(E)$ and*

$$\|T_{\sigma,\theta}(\underline{a})\|_{c\sigma}\le C\,\nu_\sigma(\underline{a}),\qquad \underline{a}\in\Gamma^s_\sigma(E).$$

Proof: The technique being well-known, we only sketch the procedure and the way bounds may be determined.

Take $k=(s-1)^{-1}>0$. From Stirling's formula, including Binet's function [4, Theorem 8.5b], and the inequality $t^d\le d^de^{t-d}$, $t,d>0$, one obtains constants $c_0(s)=c_0(k)\ge1$ and $c_1(s)=c_1(k)\ge1$ such that

$$\frac{(p!)^{1/k}}{\Gamma\left(1+\frac{p}{k}\right)}\le c_0(k)(k^{1/k}e^{2\pi})^p,\quad \frac{\Gamma\left(1+\frac{p}{k}\right)}{(p!)^{1/k}}\le c_1(k)\left(k^{-1/k}e^{2\pi}\right)^p,\quad p\in\mathbb{N}. \tag{5}$$

Given $\underline{a}=\{a_m\}_{m=0}^\infty\in\Gamma^s_\sigma(E)$, the series $\sum_{m=0}^\infty\frac{a_m}{\Gamma(1+\frac{m}{k})}z^m$ is seen to converge for $|z|<R_0=\frac{1}{\sigma k^{1/k}}$, and it defines there a holomorphic function φ; we take $R=\dfrac{1}{2e^{2\pi}}R_0<R_0$ and define the function

$$T_\sigma(\underline{a})(z)=\frac{k}{z^k}\int_0^R\varphi(t)e^{-t^k/z^k}t^{k-1}\,dt,\quad z\in S,$$

that is holomorphic from S to E (principal values are considered).

The equality

$$\frac{k}{z^k}\int_0^\infty\frac{t^p}{\Gamma\left(1+\frac{p}{k}\right)}e^{-t^k/z^k}t^{k-1}\,dt=z^p,\quad p\in\mathbb{N},$$

and the splitting

$$\varphi(z)=\sum_{p=0}^{m-1}\frac{a_p}{\Gamma\left(1+\frac{p}{k}\right)}z^p\ +\ z^m\psi_m(z),\quad m\ge1,$$

lead to the following decomposition:

$$T_\sigma(\underline{a})(z)-\sum_{p=0}^{m-1}a_pz^p=f_1(z)-f_2(z),$$

where

$$\begin{aligned}f_1(z)&=\frac{k}{z^k}\int_0^R e^{-t^k/z^k}t^{k+m-1}\psi_m(t)\,dt,\\ f_2(z)&=\frac{k}{z^k}\int_R^\infty e^{-t^k/z^k}t^{k-1}\sum_{p=0}^{m-1}\left(\frac{a_p}{\Gamma\left(1+\frac{p}{k}\right)}t^p\right)dt.\end{aligned}$$

Standard estimations, together with (5), show that for every $\theta_0\in(0,2\pi)$ with $\theta_0<(s-1)\pi$ and for every $z\in S_{\theta_0}$ we have

$$\|f_1(z)\|\le\frac{2c_0(k)c_1(k)}{\cos(k\theta_0/2)}\left(\frac{e^{4\pi}\sigma}{(\cos(k\theta_0/2))^{1/k}}\right)^m(m!)^{s-1}|z|^m\nu_\sigma(\underline{a})$$

and

$$\|f_2(z)\|\le\frac{2c_0(k)c_1(k)}{\cos(k\theta_0/2)}\left(\frac{2e^{4\pi}\sigma}{(\cos(k\theta_0/2))^{1/k}}\right)^m(m!)^{s-1}|z|^m\nu_\sigma(\underline{a}).$$

Hence, we obtain

$$\|T_\sigma(\underline{a})(z) - \sum_{p=0}^{m-1} a_p z^p\| \le \frac{4c_0(k)c_1(k)}{\cos(k\theta_0/2)}\left(\frac{2e^{4\pi}\sigma}{(\cos(k\theta_0/2))^{1/k}}\right)^m (m!)^{s-1}|z|^m \nu_\sigma(\underline{a})$$
$$= C(s,\theta_0)(c_2(s,\theta_0)\sigma)^m (m!)^{s-1}|z|^m \nu_\sigma(\underline{a}),$$

so that $T_\sigma(\underline{a}) \in \mathcal{A}^s(S,E)$, $TA(T_\sigma(\underline{a})) = \underline{a}$, $T_{\sigma,\theta_0}(\underline{a}) \in \mathcal{A}^s_{c_2\sigma}(S_{\theta_0},E)$ and $M_{c_2\sigma}(T_{\sigma,\theta_0}(\underline{a})) \le C(s,\theta_0)\nu_\sigma(\underline{a})$.

Given $\theta \in (0,2\pi)$ such that $\theta < (s-1)\pi$, it suffices to take $\theta_0 = \frac{1}{2}(\theta + \min(2\pi,(s-1)\pi))$ and apply part b) in Proposition 3.2 to conclude that there exists $c_3 = c_3(\theta_0,\theta) = c_3(s,\theta) > 1$ such that, if we put $c = c(s,\theta) := c_2(s,\theta_0)c_3(s,\theta)$ and $C = C(s,\theta) := C(s,\theta_0)$, then

$$T_{\sigma,\theta}(\underline{a}) \in \mathcal{W}^s_{c\sigma}(S_\theta,E) \quad \text{and} \quad \|T_{\sigma,\theta}(\underline{a})\|_{c\sigma} \le C\nu_\sigma(\underline{a}),$$

as desired. □

Explicit expressions for $c(s,\theta)$ and $C(s,\theta)$ may be given.

We note that one can obtain a linear continuous version of the classical Borel's theorem for Gevrey $\mathcal{C}^\infty$ functions on $\mathbb{R}$ by applying Theorem 4.1 (one or two times, according to whether $s > 2$ or $1 < s \le 2$) on sectors of suitably small opening. Although this version has already been given by Petzsche [9, Theorem 2.1], our solution has the particular feature that the extension operator provides functions analytic on $\mathbb{R} - \{0\}$.

The next result is not difficult to obtain and will be decisive when it comes to going from the one-variable problem to the several variables one.

Proposition 4.2 *Let $S_{\boldsymbol{\theta}}$ and $V_{\boldsymbol{\varphi}}$ be polysectors of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, E a Banach space, $\boldsymbol{\sigma},\boldsymbol{\tau} \in (0,\infty)^n$, and $\boldsymbol{s},\boldsymbol{t} \in (1,\infty)^n$. Then, the map*

$$f \in \mathcal{W}^{(\boldsymbol{s},\boldsymbol{t})}_{(\boldsymbol{\sigma},\boldsymbol{\tau})}(S_{\boldsymbol{\theta}} \times V_{\boldsymbol{\varphi}},E) \to f^* \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},\mathcal{W}^{\boldsymbol{t}}_{\boldsymbol{\tau}}(V_{\boldsymbol{\varphi}},E))$$

defined for every $\boldsymbol{z} \in S_{\boldsymbol{\theta}}$ by $f^*(\boldsymbol{z}) = f(\boldsymbol{z},\cdot)$ is an isomorphism. We have $\|f^*(\boldsymbol{z})\|_{\boldsymbol{\tau}} \le \|f\|_{(\boldsymbol{\sigma},\boldsymbol{\tau})}$, $\|f^*\|_{\boldsymbol{\sigma}} = \|f\|_{(\boldsymbol{\sigma},\boldsymbol{\tau})}$ and, for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\boldsymbol{\beta} \in \mathbb{N}^m$,

$$D^{(\boldsymbol{\alpha},\boldsymbol{\beta})}f(\boldsymbol{z},\boldsymbol{\omega}) = D^{\boldsymbol{\beta}}(D^{\boldsymbol{\alpha}}f^*(\boldsymbol{z}))(\boldsymbol{\omega}), \qquad (\boldsymbol{z},\boldsymbol{\omega}) \in S_{\boldsymbol{\theta}} \times V_{\boldsymbol{\varphi}}. \tag{6}$$

As a first application of this result we get the

Proof of Theorem 3.3: We apply induction on the number of variables n. The case $n=1$ has been already solved. Suppose the result holds for $n-1$ variables, $n \ge 2$, and take $\boldsymbol{\sigma} \in (0,\infty)^n$ and $\underline{a} = \{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$.

Fix $m \in \mathbb{N}$ and consider $\underline{a}_m = \{a_{(m,\boldsymbol{\beta})}\}_{\boldsymbol{\beta}\in\mathbb{N}^{1'}}$. We recall that $1' = \{2,3,\ldots,n\}$. It is clear that $\underline{a}_m \in \Gamma^{\boldsymbol{s}_{1'}}_{\boldsymbol{\sigma}_{1'}}(E)$ and

$$\nu_{\boldsymbol{\sigma}_{1'}}(\underline{a}_m) \le (m!)^{s_1-1}\sigma_1^m \nu_{\boldsymbol{\sigma}}(\underline{a}). \tag{7}$$

By the induction hypothesis, there exist $\boldsymbol{c}_{1'} = (c_2(s_2,\theta_2),\ldots,c_n(s_n,\theta_n)) \in (1,\infty)^{1'}$, $C_{1'} = C_{1'}(\boldsymbol{s},\boldsymbol{\theta}) > 0$ and a linear map

$$T_{\boldsymbol{\sigma}_{1'},\boldsymbol{\theta}_{1'}}\colon \Gamma^{\boldsymbol{s}_{1'}}_{\boldsymbol{\sigma}_{1'}}(E) \to \mathcal{W}^{\boldsymbol{s}_{1'}}_{\boldsymbol{c}_{1'}\boldsymbol{\sigma}_{1'}}(S_{\boldsymbol{\theta}_{1'}},E)$$

such that

$$\mathcal{J}(T_{\boldsymbol{\sigma}_{1'},\boldsymbol{\theta}_{1'}}(\underline{a}_m)) = \underline{a}_m, \quad \|T_{\boldsymbol{\sigma}_{1'},\boldsymbol{\theta}_{1'}}(\underline{a}_m)\|_{\boldsymbol{c}_{1'}\boldsymbol{\sigma}_{1'}} \le C_{1'}\,\nu_{\boldsymbol{\sigma}_{1'}}(\underline{a}_m).$$

Put $E_1 = \mathcal{W}^{\boldsymbol{s}_{1'}}_{\boldsymbol{c}_{1'}\boldsymbol{\sigma}_{1'}}(S_{\boldsymbol{\theta}_{1'}},E)$; taking into account (7) we see that

$$\underline{b} = \{T_{\boldsymbol{\sigma}_{1'},\boldsymbol{\theta}_{1'}}(\underline{a}_m)\}_{m=0}^\infty \in \Gamma^{s_1}_{\sigma_1}(E_1) \;\text{ and } \nu_{\sigma_1}(\underline{b}) \le C_{1'}\nu_{\boldsymbol{\sigma}}(\underline{a}).$$

The one-dimensional result ensures the existence of $c_1 = c_1(s_1,\theta_1) > 1$, $C_1 = C_1(s_1,\theta_1) > 0$ and a linear map $T_{\sigma_1,\theta_1}\colon \Gamma^{s_1}_{\sigma_1}(E_1) \to \mathcal{W}^{s_1}_{c_1\sigma_1}(S_{\theta_1},E_1)$ such that

$$\mathcal{J}(T_{\sigma_1,\theta_1}(\underline{b})) = \underline{b}, \quad \|T_{\sigma_1,\theta_1}(\underline{b})\|_{c_1\sigma_1} \le C_1\nu_{\sigma_1}(\underline{b}).$$

If we take $\boldsymbol{c} = (c_1, c_{1'})$, we have by Proposition 4.2 that $\mathcal{W}^{s_1}_{c_1\sigma_1}(S_{\theta_1}, E_1)$ and $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\theta}}, E)$ are isomorphic. We define $T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a}) \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c\sigma}}(S_{\boldsymbol{\theta}}, E)$ as the function corresponding to $T_{\sigma_1,\theta_1}(\underline{b})$ via that isomorphism, so that

$$\|T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})\|_{\boldsymbol{c\sigma}} = \|T_{\sigma_1,\theta_1}(\underline{b})\|_{c_1\sigma_1} \leq C_1\nu_{\sigma_1}(\underline{b}) \leq C_1 C_{1'}\nu_{\boldsymbol{\sigma}}(\underline{a}) = C\nu_{\boldsymbol{\sigma}}(\underline{a}).$$

Finally, observe that $TA(T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a}))$ is coherent and (6) holds; so, for every $\boldsymbol{\alpha} = (m, \boldsymbol{\beta}) \in \mathbb{N}\times\mathbb{N}^{1'}$ we may write

$$\begin{aligned}\lim_{\boldsymbol{z}\to 0} \frac{D^{\boldsymbol{\alpha}}(T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a}))(\boldsymbol{z})}{\boldsymbol{\alpha}!} &= \lim_{\boldsymbol{z}_{1'}\to 0} \frac{1}{\boldsymbol{\beta}!} D^{\boldsymbol{\beta}}\Big(\lim_{z_1\to 0} \frac{D^m(T_{\sigma_1,\theta_1}(\underline{b}))(z_1)}{m!}\Big)(\boldsymbol{z}_{1'}) \\ &= \lim_{\boldsymbol{z}_{1'}\to 0} \frac{D^{\boldsymbol{\beta}}(T_{\boldsymbol{\sigma}_{1'},\boldsymbol{\theta}_{1'}}(\underline{a}_m))(\boldsymbol{z}_{1'})}{\boldsymbol{\beta}!} = a_{(m,\boldsymbol{\beta})} = a_{\boldsymbol{\alpha}};\end{aligned}$$

as desired, we conclude that $\mathcal{J}(T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})) = \underline{a}$. □

Before proceeding to the proof of Theorem 3.4, we need some information on the behaviour of the one variable solution when it takes its values in a Banach space of the type $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$.

Let E be a Banach space, $n \geq 1$, $\boldsymbol{s} \in (1,\infty)^n$, $\boldsymbol{\sigma} \in (0,\infty)^n$ and $\boldsymbol{\theta} \in (0, 2\pi)^n$ with $\boldsymbol{\theta} < (\boldsymbol{s} - \boldsymbol{1})\pi$; let $t > 1$, $\tau > 0$ and $\rho \in (0, 2\pi)$ with $\rho < (t-1)\pi$. Suppose that for every $\mu \in \mathbb{N}$ we are given a function $f_\mu \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ in such a way that $\underline{f} = \{f_\mu\}_{\mu=0}^{\infty} \in \Gamma^t_\tau(\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E))$. Take $\ell = (t-1)^{-1}$, $R = (2e^{2\pi}\tau\ell^{1/\ell})^{-1}$; by the proof of Theorem 4.1, we know that the function $H^* = T_{\tau,\rho}(\underline{f})\colon S_\rho \to \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ given by

$$H^*(\omega) = \frac{\ell}{\omega^\ell}\int_0^R \Big(\sum_{\mu=0}^{\infty} \frac{f_\mu}{\Gamma(1+\frac{\mu}{\ell})} t^\mu\Big) e^{-t^\ell/\omega^\ell} t^{\ell-1}\,dt$$

belongs to $\mathcal{W}^t_{c(t,\rho)\tau}(S_\rho, \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E))$ for suitable $c(t,\rho) > 1$, and $\mathcal{J}(H^*) = \{f_\mu \colon \mu \in \mathbb{N}\}$. So the function $H\colon S_\rho \times S_{\boldsymbol{\theta}} \to E$ given by $H(\omega, \boldsymbol{z}) = H^*(\omega)(\boldsymbol{z})$ belongs, by Proposition 4.2, to $\mathcal{W}^{t,\boldsymbol{s}}_{(c(t,\rho)\tau,\boldsymbol{\sigma})}(S_\rho \times S_{\boldsymbol{\theta}}, E)$, and for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ we have

$$\begin{aligned}D^{(0,\boldsymbol{\alpha})}H(\omega, \boldsymbol{z}) &= D^{\boldsymbol{\alpha}}(H^*(\omega))(\boldsymbol{z}) \\ &= \frac{\ell}{\omega^\ell}\int_0^R \Big(\sum_{\mu=0}^{\infty} \frac{D^{\boldsymbol{\alpha}} f_\mu(\boldsymbol{z})}{\Gamma(1+\frac{\mu}{\ell})} t^\mu\Big) e^{-t^\ell/\omega^\ell} t^{\ell-1}\,dt. \qquad (8)\end{aligned}$$

Lemma 4.3 *If for every $m, \mu \in \mathbb{N}$ and $j \in N$ we have*

$$\lim_{\substack{z_j\to 0\\ z_j\in S_{\theta_j}}} D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z}) = 0 \qquad \textit{uniformly on } S_{\boldsymbol{\theta}_{j'}},$$

then for every $m \in \mathbb{N}$ and $j \in N$

$$\lim_{\substack{z_j\to 0\\ z_j\in S_{\theta_j}}} D^{(0,m\boldsymbol{e}_j)} H(\omega, \boldsymbol{z}) = 0 \qquad \textit{uniformly on } S_\rho \times S_{\boldsymbol{\theta}_{j'}}.$$

Proof: By (8) we have

$$\begin{aligned}D^{(0,m\boldsymbol{e}_j)}H(\omega, \boldsymbol{z}) &= D^{m\boldsymbol{e}_j}\Big(H^*(\omega)\Big)(\boldsymbol{z}) \\ &= \frac{\ell}{\omega^\ell}\int_0^R \Big(\sum_{\mu=0}^{\infty} \frac{D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z})}{\Gamma\left(1+\frac{\mu}{\ell}\right)}\, t^\mu\Big) e^{-t^\ell/\omega^\ell}\, t^{\ell-1}\,dt.\end{aligned}$$

Given $\varepsilon > 0$ there exists $\mu_0 \in \mathbb{N}$ such that for every $\mu \geq \mu_0$, every $\boldsymbol{z} \in S_{\boldsymbol{\theta}}$ and every $t \in [0, R]$ one has

$$\Big\|\sum_{\mu=\mu_0}^{\infty} \frac{D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z})}{\Gamma\left(1+\frac{\mu}{\ell}\right)}\, t^\mu\Big\| < \varepsilon.$$

Since

$$\lim_{\substack{z_j\to 0\\ z_j\in S_{\theta_j}}} D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z}) = 0, \qquad \mu = 0, 1, \ldots, \mu_0 - 1,$$

uniformly on $S_{\boldsymbol{\theta}_{j'}}$, there exists $\delta > 0$ such that whenever $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in S_{\boldsymbol{\theta}}$ and $z_j \in S_{\theta_j} \cap D_\delta(0)$ we have

$$\|D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z})\| \leq \frac{\varepsilon\,\Gamma\left(1+\frac{\mu}{\ell}\right)}{\mu_0 R^\mu}, \quad \mu = 0, 1, \ldots, \mu_0 - 1.$$

Hence, for every $\boldsymbol{z} \in S_{\boldsymbol{\theta}}$ with $z_j \in S_{\theta_j} \cap D_\delta(0)$ and every $\omega \in S_\rho$ we have

$$\begin{aligned}
\|D^{(0,m\boldsymbol{e}_j)} H(\omega, \boldsymbol{z})\| &\leq \frac{\ell}{|\omega|^\ell} \int_0^R \Bigg[\Big\| \sum_{\mu=0}^{\mu_0-1} \frac{D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z})}{\Gamma\left(1+\frac{\mu}{\ell}\right)} t^\mu \Big\| \\
&+ \Big\| \sum_{\mu=\mu_0}^{\infty} \frac{D^{m\boldsymbol{e}_j} f_\mu(\boldsymbol{z})}{\Gamma\left(1+\frac{\mu}{\ell}\right)} t^\mu \Big\| \Bigg] \exp\left(-t^\ell \Re\Big(\frac{1}{\omega^\ell}\Big)\right) t^{\ell-1}\, dt \\
&\leq 2\varepsilon \frac{\ell}{|\omega|^\ell} \int_0^R \exp\left(-t^\ell \Re\Big(\frac{1}{\omega^\ell}\Big)\right) t^{\ell-1}\, dt \\
&\leq 2\varepsilon \frac{\ell}{|\omega|^\ell} \int_0^\infty \exp\left(-t^\ell \frac{\cos(\ell\rho/2)}{|\omega|^\ell}\right) t^{\ell-1}\, dt \\
&= \frac{2\varepsilon}{\cos(\ell\rho/2)} \frac{\ell}{|\omega|^\ell} \int_0^\infty \exp\left(\frac{-u^\ell}{|\omega|^\ell}\right) u^{\ell-1}\, du = \frac{2\varepsilon}{\cos(\ell\rho/2)},
\end{aligned}$$

and the proof is complete. □

Proof of Theorem 3.4:

Let $\mathcal{G} = \{f_{jm} : j \in N,\ m \in \mathbb{N}\} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ be given; we know that $\mathcal{G}_1 = \{f_{1m}\}_{m=0}^\infty \in \Gamma^{s_1}_{\sigma_1}(\mathcal{W}^{\boldsymbol{s}_{1'}}_{\boldsymbol{\sigma}_{1'}}(S_{\boldsymbol{\theta}_{1'}}, E))$. By Theorem 4.1, there exist $c_1 = c_1(s_1, \theta_1) > 1$ and $C_1 = C_1(s_1, \theta_1) > 0$ such that

$$T_{\sigma_1,\theta_1}(\mathcal{G}_1) = H_1^{[1]*} \in \mathcal{W}^{s_1}_{c_1\sigma_1}(S_{\theta_1}, \mathcal{W}^{\boldsymbol{s}_{1'}}_{\boldsymbol{\sigma}_{1'}}(S_{\boldsymbol{\theta}_{1'}}, E))$$

and $H_1^{[1]*} \sim \sum_{m=0}^\infty f_{1m} z_1^m$, $\|H_1^{[1]*}\|_{c_1\sigma_1} \leq C_1\, \nu_{\sigma_1}(\mathcal{G}_1)$.

By Proposition 4.2 the function $H^{[1]}$ given by $H^{[1]}(\boldsymbol{z}) = H_1^{[1]*}(z_1)(\boldsymbol{z}_{1'})$ belongs to $\mathcal{W}^{\boldsymbol{s}}_{(c_1\sigma_1, \boldsymbol{\sigma}_{1'})}(S_{\boldsymbol{\theta}}, E)$, and $\|H^{[1]}\|_{(c_1\sigma_1,\boldsymbol{\sigma}_{1'})} = \|H_1^{[1]*}\|_{c_1\sigma_1}$. Put $\mathcal{J}_1(H^{[1]}) = \{h^{[1]}_{jm}\}$; for every $\boldsymbol{z}_{1'} \in S_{\boldsymbol{\theta}_{1'}}$ we have

$$\begin{aligned}
h^{[1]}_{1m}(\boldsymbol{z}_{1'}) &= \lim_{\substack{z_1\to 0\\ z_1\in S_{\theta_1}}} \frac{D^{m\boldsymbol{e}_1} H^{[1]}(\boldsymbol{z})}{m!} \\
&= \lim_{\substack{z_1\to 0\\ z_1\in S_{\theta_1}}} \frac{(H_1^{[1]*})^{m)}(z_1)(\boldsymbol{z}_{1'})}{m!} = f_{1m}(\boldsymbol{z}_{1'}).
\end{aligned}$$

Let us consider the function $H_2^{[1]*}$ given by

$$H_2^{[1]*}(z_2)(\boldsymbol{z}_{2'}) = H^{[1]}(z_2, \boldsymbol{z}_{2'}), \qquad z_2 \in S_{\theta_2},\ \boldsymbol{z}_{2'} \in S_{\boldsymbol{\theta}_{2'}}.$$

By Proposition 4.2, $H_2^{[1]*} \in \mathcal{W}^{s_2}_{\sigma_2}(S_{\theta_2}, \mathcal{W}_{(c_1\sigma_1, \boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}_{2'}}, E))$ and

$$H_2^{[1]*} \sim \sum_{m=0}^\infty h^{[1]}_{2m}\, z_2^m.$$

From the coherence conditions for $\mathcal{G}$ and $\mathcal{J}_1(H^{[1]})$ we have that for every $m, k \in \mathbb{N}$

$$\lim_{\substack{z_1\to 0\\ z_1\in S_{\theta_1}}} \frac{D^{m\boldsymbol{e}_1}(f_{2k} - h^{[1]}_{2k})(\boldsymbol{z}_{2'})}{m!} = \lim_{\substack{z_2\to 0\\ z_2\in S_{\theta_2}}} \frac{D^{k\boldsymbol{e}_2}(f_{1m} - h^{[1]}_{1m})(\boldsymbol{z}_{1'})}{k!} = 0.$$

Since $\mathcal{G} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, we have $\mathcal{G}_2 = \{f_{2m}\}_{m=0}^\infty \in \Gamma^{s_2}_{\sigma_2}(\mathcal{W}^{\boldsymbol{s}_{2'}}_{\boldsymbol{\sigma}_{2'}}(S_{\boldsymbol{\theta}_{2'}}, E))$; on the other hand, $\{h^{[1]}_{2m}\}_{m=0}^\infty \in \Gamma^{s_2}_{\sigma_2}(\mathcal{W}^{\boldsymbol{s}_{2'}}_{(c_1\sigma_1, \boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}_{2'}}, E))$. So,

$$\{f_{2m} - h^{[1]}_{2m}\}_{m=0}^\infty \in \Gamma^{s_2}_{\sigma_2}(\mathcal{W}^{\boldsymbol{s}_{2'}}_{(c_1\sigma_1, \boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}_{2'}}, E)),$$

and we can apply Theorem 4.1 to obtain constants $c_2 = c_2(s_2, \theta_2) > 1$ and $C_2 = C_2(s_2, \theta_2) > 0$ such that the function

$$T_{\sigma_2,\theta_2}(\{f_{2m} - h^{[1]}_{2m}\}_{m=0}^{\infty}) = H_2^{[2]*} \in \mathcal{W}^{s_2}_{c_2\sigma_2}(S_{\theta_2}, \mathcal{W}^{\boldsymbol{s}_{2'}}_{(c_1\sigma_1,\boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}_{2'}}, E)),$$

and

$$H_2^{[2]*} \sim \sum_{m=0}^{\infty} (f_{2m} - h^{[1]}_{2m}) z_2^m, \quad \|H_2^{[2]*}\|_{c_2\sigma_2} \leq C_2\, \nu_{\sigma_2}(\{f_{2m} - h^{[1]}_{2m}\}_{m=0}^{\infty}). \tag{9}$$

Proposition 4.2 implies that $H^{[2]}$, given by $H^{[2]}(\boldsymbol{z}) = H_2^{[2]*}(z_2)(\boldsymbol{z}_{2'})$, belongs to $\mathcal{W}^{\boldsymbol{s}}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}}, E)$, and $\|H^{[2]}\|_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2\}'})} = \|H_2^{[2]*}\|_{c_2\sigma_2}$. Observe that if we put $\mathcal{J}_1(H^{[2]}) = \{h^{[2]}_{jm}\}$, by the previous Lemma and (9) we have

$$h^{[2]}_{1m} = 0, \quad h^{[2]}_{2m} = f_{2m} - h^{[1]}_{2m}, \quad m \in \mathbb{N}.$$

So, $F^{[2]} = H^{[1]} + H^{[2]}$ is in $\mathcal{W}^{\boldsymbol{s}}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2\}'})}(S_{\boldsymbol{\theta}}, E)$ and, if $\mathcal{J}_1(F^{[2]}) = \{f^{[2]}_{jm}\}$, we have $f^{[2]}_{jm} = f_{jm}$ for $j = 1, 2$ and $m \in \mathbb{N}$. Consider the function $F_3^{[2]*}$ defined by

$$F_3^{[2]*}(z_3)(\boldsymbol{z}_{3'}) = F^{[2]}(z_3, \boldsymbol{z}_{3'}), \qquad z_3 \in S_{\theta_3},\ \boldsymbol{z}_{3'} \in S_{\boldsymbol{\theta}_{3'}}.$$

According to Proposition 4.2, $F_3^{[2]*} \in \mathcal{W}^{s_3}_{\sigma_3}(S_{\theta_3}, \mathcal{W}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}_{3'}}, E))$ and

$$F_3^{[2]*} \sim \sum_{m=0}^{\infty} f^{[2]}_{3m}\, z_3^m.$$

From the coherence conditions for $\mathcal{G}$ and $\mathcal{J}_1(F^{[2]})$ we have that for every $m, k \in \mathbb{N}$ and for $j = 1, 2$,

$$\lim_{\substack{z_j \to 0 \\ z_j \in S_{\theta_j}}} \frac{D^{m\boldsymbol{e}_j}(f_{3k} - f^{[2]}_{3k})(\boldsymbol{z}_{3'})}{m!} = \lim_{\substack{z_3 \to 0 \\ z_3 \in S_{\theta_3}}} \frac{D^{k\boldsymbol{e}_3}(f_{jm} - f^{[2]}_{jm})(\boldsymbol{z}_{j'})}{k!} = 0.$$

As $\mathcal{G} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, we have $\mathcal{G}_3 = \{f_{3m}\}_{m=0}^{\infty} \in \Gamma^{s_3}_{\sigma_3}(\mathcal{W}^{\boldsymbol{s}_{3'}}_{\boldsymbol{\sigma}_{3'}}(S_{\boldsymbol{\theta}_{3'}}, E))$; on the other hand, $\{f^{[3]}_{3m}\}_{m=0}^{\infty} \in \Gamma^{s_3}_{\sigma_3}(\mathcal{W}^{\boldsymbol{s}_{3'}}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}_{3'}}, E))$, so that

$$\{f_{3m} - f^{[2]}_{3m}\}_{m=0}^{\infty} \in \Gamma^{s_3}_{\sigma_3}(\mathcal{W}^{\boldsymbol{s}_{3'}}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}_{3'}}, E)).$$

We can apply Theorem 4.1 and obtain constants $c_3 = c_3(s_3, \theta_3) > 1$ and $C_3 = C_3(s_3, \theta_3) > 0$ such that

$$T_{\sigma_3,\theta_3}(\{f_{3m} - f^{[2]}_{3m}\}_{m=0}^{\infty}) = H_3^{[3]*} \in \mathcal{W}^{s_3}_{c_3\sigma_3}(S_{\theta_3}, \mathcal{W}^{\boldsymbol{s}_{3'}}_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}_{3'}}, E)),$$

and

$$H_3^{[3]*} \sim \sum_{m=0}^{\infty} (f_{3m} - f^{[2]}_{3m}) z_3^m, \quad \|H_3^{[3]*}\|_{c_3\sigma_3} \leq C_3\, \nu_{\sigma_3}(\{f_{3m} - f^{[2]}_{3m}\}_{m=0}^{\infty}). \tag{10}$$

Again by Proposition 4.2, the function $H^{[3]}$ given by $H^{[3]}(\boldsymbol{z}) = H_3^{[3]*}(z_3)(\boldsymbol{z}_{3'})$ verifies

$$H^{[3]} \in \mathcal{W}^{\boldsymbol{s}}_{(c_1\sigma_1,c_2\sigma_2,c_3\sigma_3,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}}, E),\ \|H^{[3]}\|_{(c_1\sigma_1,c_2\sigma_2,c_3\sigma_3,\boldsymbol{\sigma}_{\{1,2,3\}'})} = \|H_3^{[3]*}\|_{c_3\sigma_3}.$$

If $\mathcal{J}_1(H^{[3]}) = \{h^{[3]}_{jm}\}$, then the previous Lemma and (10) imply

$$\begin{aligned} h^{[3]}_{jm} &= 0, \quad j = 1, 2,\ m \in \mathbb{N}; \\ h^{[3]}_{3m} &= f_{3m} - f^{[2]}_{3m}, \quad m \in \mathbb{N}. \end{aligned}$$

So, $F^{[3]} = F^{[2]} + H^{[3]}$ belongs to $\mathcal{W}^{\boldsymbol{s}}_{(c_1\sigma_1,c_2\sigma_2,c_3\sigma_3,\boldsymbol{\sigma}_{\{1,2,3\}'})}(S_{\boldsymbol{\theta}},E)$, and if we put $\mathcal{J}_1(F^{[3]}) = \{f^{[3]}_{jm}\}$, we have $f^{[3]}_{jm} = f_{jm}$ for $j = 1,2,3$ and $m \in \mathbb{N}$. After the necessary steps, we would obtain a function $F = F^{[n]} = U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})$ solving the problem. Indeed, the construction shows $U_{\boldsymbol{\sigma},\boldsymbol{\theta}}$ is linear and sends $G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$ into $\mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c}\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E)$, with $\boldsymbol{c} = (c_1(s_1,\theta_1),\dots,c_n(s_n,\theta_n)) \in (1,\infty)^n$; moreover, $\mathcal{J}_1(U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})) = \mathcal{G}$. Finally, we can write

$$
\begin{aligned}
\|H^{[1]}\|_{\boldsymbol{c}\boldsymbol{\sigma}} &\le \|H^{[1]}\|_{(c_1\sigma_1,\boldsymbol{\sigma}_{1'})} = \|H^{[1]*}_1\|_{c_1\sigma_1} \\
&\le C_1\,\nu_{\sigma_1}(\mathcal{G}_1) \le C_1 N_{\boldsymbol{\sigma}}(\mathcal{G}); \\
\|H^{[2]}\|_{\boldsymbol{c}\boldsymbol{\sigma}} &\le \|H^{[2]}\|_{(c_1\sigma_1,c_2\sigma_2,\boldsymbol{\sigma}_{\{1,2\}'})} = \|H^{[2]*}_2\|_{c_2\sigma_2} \\
&\le C_2\,\nu_{\sigma_2}(\{f_{2m} - h^{[1]}_{2m}\}_{m=0}^{\infty}) \le C_2(\nu_{\sigma_2}(\mathcal{G}_2) + \nu_{\sigma_2}(\mathcal{J}(H^{[1]*}_2))) \\
&\le C_2(N_{\boldsymbol{\sigma}}(\mathcal{G}) + \|H^{[1]}\|_{(c_1\sigma_1,\boldsymbol{\sigma}_{1'})}) \le (1+C_1)C_2 N_{\boldsymbol{\sigma}}(\mathcal{G});
\end{aligned}
$$

and inductively,

$$
\|H^{[j]}\|_{(c_1\sigma_1,\dots,c_j\sigma_j,\boldsymbol{\sigma}_{\{1,\dots,j\}'})} \le C_j \prod_{k=1}^{j-1}(1+C_k)\,N_{\boldsymbol{\sigma}}(\mathcal{G}).
$$

Hence, we have

$$
\begin{aligned}
\|F^{[j]}\|_{(c_1\sigma_1,\dots,c_j\sigma_j,\boldsymbol{\sigma}_{\{1,\dots,j\}'})} &= \|\sum_{k=1}^{j} H^{[k]}\|_{(c_1\sigma_1,\dots,c_j\sigma_j,\boldsymbol{\sigma}_{\{1,\dots,j\}'})} \\
&\le \sum_{k=1}^{j} \|H^{[k]}\|_{(c_1\sigma_1,\dots,c_k\sigma_k,\boldsymbol{\sigma}_{\{1,\dots,k\}'})} \\
&\le \sum_{k=1}^{j} C_k \prod_{\ell=1}^{k-1}(1+C_\ell)\,N_{\boldsymbol{\sigma}}(\mathcal{G}) \\
&= \Big(\prod_{k=1}^{j}(1+C_k)\,-1\Big)\,N_{\boldsymbol{\sigma}}(\mathcal{G}),
\end{aligned}
$$

and, in particular,

$$
\begin{aligned}
\|U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})\|_{\boldsymbol{c}\boldsymbol{\sigma}} &= \|F^{[n]}\|_{\boldsymbol{c}\boldsymbol{\sigma}} \\
&\le \Big(\prod_{k=1}^{n}(1+C_k)\,-1\Big)\,N_{\boldsymbol{\sigma}}(\mathcal{G}) = C\,N_{\boldsymbol{\sigma}}(\mathcal{G}),
\end{aligned}
$$

where $C = C(\boldsymbol{s},\boldsymbol{\theta}) > 0$, as desired. □

5 Rigidity properties

In order to study the rigidity of the operator $T_{\boldsymbol{\sigma},\boldsymbol{\theta}}$ constructed in Theorem 3.3, we will depart from the same setting as Thilliez [13, §2].

Let $n \ge 1$, $\boldsymbol{s} \in (1,\infty)^n$, $\boldsymbol{\sigma} \in (0,\infty)^n$, and define

$$
\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E) = \{\underline{a} = \{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n} : a_{\boldsymbol{\alpha}} \in E,\ \lim_{|\boldsymbol{\alpha}|\to\infty} \frac{\|a_{\boldsymbol{\alpha}}\|}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}-\boldsymbol{1}}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}} = 0\}.
$$

Obviously we have $\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E) \subset \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$, and $(\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E),\nu_{\boldsymbol{\sigma}})$ is a Banach space. For $\boldsymbol{\theta} \in (0,2\pi)^n$ with $\boldsymbol{\theta} < (\boldsymbol{s}-\boldsymbol{1})\pi$, consider

$$
H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E) = \{f \in \mathcal{W}^{\boldsymbol{s}}_{\boldsymbol{c}\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}},E) : \mathcal{J}(f) \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E)\},
$$

where $\boldsymbol{c} = (c_1(s_1,\theta_1),\cdots,c_n(s_n,\theta_n)) \in (1,\infty)^n$ is the constant vector in Theorem 3.3; on this space we define the norm

$$
\|f\|^{\wedge}_{\boldsymbol{\sigma}} = \|f\|_{\boldsymbol{c}\boldsymbol{\sigma}} + \nu_{\boldsymbol{\sigma}}(\mathcal{J}(f)).
$$

$(H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E), \|\cdot\|^{\wedge}_{\boldsymbol{\sigma}})$ is a Banach space, and its subspace

$$F^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) = \{f \in H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \colon \mathcal{J}(f) = \underline{0}\}$$

is closed, so that the quotient space $L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) = H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)/F^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ is a Banach space, with norm

$$|\dot{f}|_{\boldsymbol{\sigma}} = \inf_{g \in F^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)} \|f+g\|^{\wedge}_{\boldsymbol{\sigma}} = \nu_{\boldsymbol{\sigma}}(\mathcal{J}(f)) + \inf_{g \in F^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)} \|f+g\|_{\boldsymbol{c\sigma}}.$$

Let $\pi_{\boldsymbol{\sigma}} \colon H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ be the canonical map. It is clear that the map $\mathcal{J}$ induces a well-defined map $\dot{\mathcal{J}} \colon L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E)$, which indeed is an isomorphism, with inverse $\pi_{\boldsymbol{\sigma}} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}$. For every $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E)$ we have

$$\begin{aligned} |\dot{\mathcal{J}}^{-1}\underline{a}|_{\boldsymbol{\sigma}} &= |\pi_{\boldsymbol{\sigma}} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})|_{\boldsymbol{\sigma}} \le \|T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})\|^{\wedge}_{\boldsymbol{\sigma}} \\ &= \|T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})\|_{\boldsymbol{c\sigma}} + \nu_{\boldsymbol{\sigma}}(\mathcal{J} T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})) \le (1+C)\nu_{\boldsymbol{\sigma}}(\underline{a}), \end{aligned}$$

so that $\|\dot{\mathcal{J}}^{-1}\| \le 1+C$, where $C = C(\boldsymbol{s}, \boldsymbol{\theta}) \in (0, \infty)$ is the constant in Theorem 3.3. The map $P_{\boldsymbol{\sigma}} \colon H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to H^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$, defined by $P_{\boldsymbol{\sigma}} = T_{\boldsymbol{\sigma},\boldsymbol{\theta}} \circ \mathcal{J}$, is linear continuous.

Suppose that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ we choose $\boldsymbol{z}_{\boldsymbol{\alpha}} = (z^{(1)}_{\boldsymbol{\alpha}}, \ldots, z^{(n)}_{\boldsymbol{\alpha}}) \in S_{\boldsymbol{\theta}}$, and consider the map $\dot{\mathcal{K}} \colon L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to E^{\mathbb{N}^n}$ given by

$$\dot{\mathcal{K}}(\dot{f}) = \{\frac{1}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}}(P_{\boldsymbol{\sigma}} f)(\boldsymbol{z}_{\boldsymbol{\alpha}})\}_{\boldsymbol{\alpha} \in \mathbb{N}^n}.$$

$\dot{\mathcal{K}}$ is well-defined. Under suitable conditions on the points $\boldsymbol{z}_{\boldsymbol{\alpha}}$, we will prove that $\dot{\mathcal{K}}$ is "sufficiently close" to $\dot{\mathcal{J}}$.

Lemma 5.1 *Suppose there exists $k \in (0,1)$ such that*

$$\sum_{j=1}^{n} |z^{(j)}_{\boldsymbol{\alpha}}|(\alpha_j+1)^{s_j} c_j \sigma_j \le \frac{k}{C(1+C)\boldsymbol{c}^{\boldsymbol{\alpha}}}, \qquad \boldsymbol{\alpha} \in \mathbb{N}^n.$$

Then, the range of $\dot{\mathcal{K}}$ is contained in $\Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$, and $\dot{\mathcal{K}}$ admits a continuous inverse.

Proof: Fix $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $\dot{f} \in L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$. The $\boldsymbol{\alpha}$-coordinate of $\dot{\mathcal{J}}(\dot{f})$ equals $\frac{1}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}}(P_{\boldsymbol{\sigma}} f)(\boldsymbol{0})$; so, for the distance $d_{\boldsymbol{\alpha}}$ between it and the $\boldsymbol{\alpha}$-coordinate of $\dot{\mathcal{K}}(\dot{f})$ we can write

$$\begin{aligned} d_{\boldsymbol{\alpha}} &= \frac{1}{\boldsymbol{\alpha}!}\|D^{\boldsymbol{\alpha}}(P_{\boldsymbol{\sigma}} f)(\boldsymbol{z}_{\boldsymbol{\alpha}}) - D^{\boldsymbol{\alpha}}(P_{\boldsymbol{\sigma}} f)(\boldsymbol{0})\| \\ &= \frac{1}{\boldsymbol{\alpha}!}\|\int_0^1 \sum_{j=1}^{n} D^{\boldsymbol{\alpha}+\boldsymbol{e}_j}(P_{\boldsymbol{\sigma}} f)(t\boldsymbol{z}_{\boldsymbol{\alpha}}) z^{(j)}_{\boldsymbol{\alpha}}\, dt\| \\ &\le \frac{\|P_{\boldsymbol{\sigma}} f\|_{\boldsymbol{c\sigma}}}{\boldsymbol{\alpha}!} \sum_{j=1}^{n} |z^{(j)}_{\boldsymbol{\alpha}}|(\boldsymbol{\alpha}+\boldsymbol{e}_j)!^{\boldsymbol{s}}(\boldsymbol{c\sigma})^{\boldsymbol{\alpha}+\boldsymbol{e}_j} \\ &\le \frac{1}{\boldsymbol{\alpha}!} C(\boldsymbol{s}, \boldsymbol{\theta})\nu_{\boldsymbol{\sigma}}(\mathcal{J} f)\boldsymbol{\alpha}!^{\boldsymbol{s}}(\boldsymbol{c\sigma})^{\boldsymbol{\alpha}} \sum_{j=1}^{n} |z^{(j)}_{\boldsymbol{\alpha}}|(\alpha_j+1)^{s_j} c_j \sigma_j \\ &\le C\boldsymbol{c}^{\boldsymbol{\alpha}}\Big(\sum_{j=1}^{n} |z^{(j)}_{\boldsymbol{\alpha}}|(\alpha_j+1)^{s_j} c_j \sigma_j\Big)\boldsymbol{\alpha}!^{\boldsymbol{s}-\boldsymbol{1}}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}|\dot{f}|_{\boldsymbol{\sigma}}. \end{aligned}$$

By the hypotheses we find that $d_{\boldsymbol{\alpha}} \le (1+C)^{-1}k(\boldsymbol{\alpha}!)^{\boldsymbol{s}-\boldsymbol{1}}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}|\dot{f}|_{\boldsymbol{\sigma}}$, so that

$$\nu_{\boldsymbol{\sigma}}(\dot{\mathcal{K}}\dot{f} - \dot{\mathcal{J}}\dot{f}) = \sup_{\boldsymbol{\alpha} \in \mathbb{N}^n} \frac{d_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha}!)^{\boldsymbol{s}-\boldsymbol{1}}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}} \le \frac{k}{1+C}|\dot{f}|_{\boldsymbol{\sigma}}.$$

It is now clear that $\dot{\mathcal{K}}(L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)) \subset \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$, and that $\dot{\mathcal{K}} \colon L^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E) \to \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(E)$ is linear continuous with

$$\|\dot{\mathcal{K}} - \dot{\mathcal{J}}\| \le \frac{k}{1+C} < \frac{1}{1+C} \le \frac{1}{\|\dot{\mathcal{J}}^{-1}\|}.$$

The Banach isomorphism theorem lets us conclude that $\dot{\mathcal{K}}$ admits a continuous inverse. $\square$
We now give a result about the rigidity of the extension operators $T_{\boldsymbol{\sigma},\boldsymbol{\theta}}$.

Theorem 5.2 *Suppose there exists* $k \in (0,1)$ *such that*

$$\sum_{j=1}^{n} |z_{\boldsymbol{\alpha}}^{(j)}|(\alpha_j+1)^{s_j} c_j \sigma_j \leq \frac{k}{C(1+C)\boldsymbol{c}^{\boldsymbol{\alpha}}}, \qquad \boldsymbol{\alpha} \in \mathbb{N}^n.$$

If $\underline{a} \in \Gamma^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(E)$ *is such that* $D^{\boldsymbol{\alpha}}(T_{\boldsymbol{\sigma},\boldsymbol{\theta}}\underline{a})(\boldsymbol{z}_{\boldsymbol{\alpha}}) = 0$ *for all* $\boldsymbol{\alpha} \in \mathbb{N}^n$*, then* $\underline{a} = \underline{0}$.

Proof: Observe that the conditions on $\underline{a}$ amount to $\dot{\mathcal{K}}(\pi_{\boldsymbol{\sigma}} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})) = \underline{0}$; by the previous lemma, this implies $\pi_{\boldsymbol{\sigma}} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a}) = \dot{0}$, and so

$$\underline{a} = \mathcal{J} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a}) = \dot{\mathcal{J}}(\pi_{\boldsymbol{\sigma}} \circ T_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\underline{a})) = \underline{0}\,,$$

as desired. $\square$
When $n = 1$ the theorem reads as follows.

Theorem 5.3 *Let* $s > 1$, $\sigma > 0$ *and* $\theta \in (0, 2\pi)$ *with* $\theta < (s-1)\pi$. *Suppose we are given a sequence* $\{z_m\}_{m=0}^{\infty}$ *of points in* S_θ *such that there exists* $k \in (0,1)$ *with*

$$|z_m| \leq \frac{k}{\sigma C(1+C)(m+1)^s c^{m+1}}, \qquad m \in \mathbb{N}.$$

Then, if $\underline{a} \in \Gamma^{s}_{\sigma,0}(E)$ *is such that* $(T_{\sigma,\theta}\underline{a})^{(m)}(z_m) = 0$ *for every* $m \in \mathbb{N}$*, we have* $\underline{a} = \underline{0}$.

As a consequence we obtain

Corollary 5.4 *Let* s, σ, θ *and* $\{z_m\}_{m=0}^{\infty}$ *be as above, and* $\varphi \in [(s-1)\pi, 2\pi)$. *Suppose that a function* $f \in \mathcal{W}^s_\sigma(S_\varphi, E)$ *verifies*

$$\mathcal{J}f \in \Gamma^s_{\sigma,0}(E) \qquad \textit{and} \qquad (T_{\sigma,\theta}\mathcal{J}f)^{(m)}(z_m) = 0 \quad \textit{for every } m \in \mathbb{N}.$$

Then, we have $f \equiv 0$.

Proof: By the Hahn-Banach Theorem, if suffices to prove that $\phi \circ f \equiv 0$ for every $\phi \in E'$. The previous result implies $\mathcal{J}f = \underline{0}$, so that the same holds for the complex function $\phi \circ f \in \mathcal{W}^s_\sigma(S_\varphi, \mathbb{C})$. The Watson's lemma implies $\phi \circ f \equiv 0$. $\square$

We come to the study of the rigidity of the operators $U_{\boldsymbol{\sigma},\boldsymbol{\theta}}$. Let $n > 1$, $\boldsymbol{s} \in (1,\infty)^n$, $\boldsymbol{\sigma} \in (0,\infty)^n$, and $\boldsymbol{\theta} \in (0, 2\pi)^n$ with $\boldsymbol{\theta} < (\boldsymbol{s}-\boldsymbol{1})\pi$. Define $G^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(S_{\boldsymbol{\theta}}, E)$ as the set consisting of the families $\mathcal{G} = \{f_{jm}\} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma}}(S_{\boldsymbol{\theta}}, E)$ such that

$$\lim_{m\to\infty} \frac{\|f_{jm}\|_{\boldsymbol{\sigma}_{N-\{j\}}}}{m!^{s_j-1}\sigma_j^m} = 0, \quad j = 1, 2, \ldots, n;$$

equivalently, $\mathcal{G} = \{f_{jm}\} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(S_{\boldsymbol{\theta}}, E)$ if and only if $\mathcal{G}$ is a coherent first order family and, for every $j = 1, 2, \ldots, n$, we have

$$\mathcal{G}_j = \{f_{jm}\}_{m=0}^{\infty} \in \Gamma^{s_j}_{\sigma_j,0}(\mathcal{W}^{\boldsymbol{s}_{j'}}_{\boldsymbol{\sigma}_{j'}}(S_{\boldsymbol{\theta}_{j'}}, E)).$$

In the following result we use the same notation as that in the statement and proof of Theorem 3.4.

Theorem 5.5 *Let* $\mathcal{G} \in G^{\boldsymbol{s}}_{\boldsymbol{\sigma},0}(S_{\boldsymbol{\theta}}, E)$*, and let* $H^{[1]}, H^{[2]}, \ldots, H^{[n]}$ *be the successive functions obtained on constructing* $U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{G})$. *Suppose there exists a set of complex numbers* $\{z_{jm}\}_{j\in N, m\in\mathbb{N}}$ *such that:*

(i) *For every* $j \in N$ *and every* $m \in \mathbb{N}$, $z_{jm} \in S_{\theta_j}$.

(ii) *There exists* $k \in (0,1)$ *with*

$$|z_{jm}| \leq \frac{k}{\sigma_j C_j(1+C_j)(m+1)^{s_j} c_j^{m+1}}, \qquad j \in N,\ m \in \mathbb{N}.$$

(iii) *For every* j *and* m, $D^{me_j}(H^{[j]})(z_{jm}, \cdot)$ *is identically zero on* $S_{\boldsymbol{\theta}_{j'}}$.

Then, $\mathcal{G}$ *is the null family.*

Proof: Recall that $H^{[1]}(\boldsymbol{z}) = H_1^{[1]*}(z_1)(\boldsymbol{z}_{1'})$, $\boldsymbol{z} \in S_{\boldsymbol{\theta}}$, where $H_1^{[1]*} = T_{\sigma_1,\theta_1}(\mathcal{G}_1)$. For $j = 1$, and according to (6) in Proposition 4.2, condition (iii) means that $(H_1^{[1*]})^{m)}(z_{1m}) = 0$. Conditions (i) and (ii) allow then to apply Theorem 5.3 to conclude that $\mathcal{G}_1$ is the null family. T_{σ_1,θ_1} being linear, we have $H_1^{[1]*} = 0$, and also $H^{[1]} = 0 = H_2^{[1]*}$, so that we have in fact $h_{2m}^{[1]} = 0$ for every m. Then,

$$\{f_{2m} - h_{2m}^{[1]}\}_{m=0}^{\infty} = \{f_{2m}\}_{m=0}^{\infty} = \mathcal{G}_2,$$

and $H_2^{[2]*} = T_{\sigma_2,\theta_2}(\mathcal{G}_2)$; we may repeat the preceding argument to obtain $\mathcal{G}_2$ is the null family, and so on. □

The following lemma will readily lead us to an easy corollary of the previous result.

Lemma 5.6 *Let $\boldsymbol{s}$ and $\boldsymbol{\sigma}$ be as usual, and let $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n) \in (0, 2\pi)^n$ be such that there is $j \in N$ with $\varphi_j \geq (s_j - 1)\pi$. If $f \in \mathcal{W}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(S_{\boldsymbol{\varphi}}, E)$ and $\mathcal{J}_1(f)$ is the null family, then f is identically zero on $S_{\boldsymbol{\varphi}}$.*

Proof: It suffices to fix an arbitrary element $\boldsymbol{z}_{j'} \in S_{\boldsymbol{\varphi}_{j'}}$, and show that the function $f(\cdot, \boldsymbol{z}_{j'})$ admits the null series as s_j-Gevrey asymptotic expansion at 0 following S_{φ_j}. The opening of this sector being large, we may apply Watson's lemma to conclude that this function is identically zero. □

Corollary 5.7 *Let $\boldsymbol{s}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ be as usual, and let $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n) \in (0, 2\pi)^n$ be such that $\boldsymbol{\theta} < \boldsymbol{\varphi}$ and there exists $j \in N$ with $\varphi_j \geq (s_j - 1)\pi$. Suppose $f \in \mathcal{W}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(S_{\boldsymbol{\varphi}}, E)$ verifies:*

a) Its restriction to $S_{\boldsymbol{\theta}}$, say $\tilde{f}$, is such that $\mathcal{J}_1(\tilde{f}) \in G_{\boldsymbol{\sigma},0}^{\boldsymbol{s}}(S_{\boldsymbol{\theta}}, E)$.

b) Let $H^{[1]}, H^{[2]}, \ldots, H^{[n]}$ be the successive functions obtained on constructing $U_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\mathcal{J}_1(\tilde{f}))$. There exists a set of complex numbers $\{z_{jm}\}_{j\in N, m\in\mathbb{N}}$ verifying conditions (i), (ii) *and* (iii) *in Theorem 5.5.*
Then, f is identically zero.

References

[1] W. Balser, *From divergent power series to analytic functions. Theory and application of multisummable power series*, Lecture Notes in Math. 1582, Springer-Verlag, Berlín, 1994.

[2] J. Chaumat, A.M. Chollet, Théorème de Whitney dans des classes ultradifférentiables, *C. R. Acad. Sci. Paris* **315** (1992), 901–906.

[3] Y. Haraoka, Theorems of Sibuya-Malgrange type for Gevrey functions of several variables, *Funkcial. Ekvac.* **32** (1989), 365–388.

[4] P. Henrici, *Applied and Computational Complex Analysis*, Vol. II, John Wiley and Sons, Inc., 1977.

[5] J. A. Hernández, Desarrollos asintóticos en polisectores. Problemas de existencia y unicidad, Ph. D. Thesis, University of Valladolid, 1994.

[6] H. Majima, Analogues of Cartan's Decomposition Theorem in Asymptotic Analysis, *Funkcial. Ekvac.* **26** (1983), 131–154.

[7] H. Majima, *Asymptotic Analysis for Integrable Connections with Irregular Singular Points*, Lecture Notes in Math. 1075, Springer, Berlín, 1984.

[8] J. Martinet, J.P. Ramis, Problèmes de modules pour des équations différentielles non linéaires du premier ordre, *Publ. I.H.E.S.* **55** (1982), 63–164.

[9] H.J. Petzsche, On E. Borel's theorem, *Math. Ann.* **282** (1988), 299–313.

[10] J.P. Ramis, Dévissage Gevrey, *Asterisque* **59–60** (1978), 173–204.

[11] J.P. Ramis, *Les séries k-sommables et leurs applications*, Lecture Notes in Phys. 126, Springer-Verlag, Berlín, 1980.

[12] J. Sanz, F. Galindo, On strongly asymptotically developable functions and the Borel-Ritt theorem, *Studia Math.* **133** (1999), 231–248.

[13] V. Thilliez, Extension Gevrey et rigidité dans un secteur, *Studia Math.* **117** (1995), 29–41.