A generalized moment problem for rapidly decreasing smooth vector functions of several variables

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Abstract: Generalized moments may be defined for functions of several variables. A theorem is proved characterizing the families of functions which are generalized moments of a smooth rapidly decreasing function.

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1 Introduction

Plenty of variants and generalizations of the moment problem, initially posed and solved by Stieltjes [11], have been the object of study for over a century. One of these results was obtained by Durán [1]: given an arbitrary sequence $\{a_m\}_{m=0}^{\infty}$ of complex numbers, there exists a function f belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth (complex) functions, with support in $[0,\infty)$ and such that

$$\int_0^{\infty} f(x)x^m\,dx = a_m, \qquad m \in \mathbb{N} = \{0,1,2,\ldots\}.$$

The integral on the left is the *moment of order* m of f, and we will write $M(f)$ for the set of moments of f.

Let E be a complex Fréchet space and V be an open subset of $\mathbb{R}^n$. We denote $\mathcal{S}(V,E)$ the space of rapidly decreasing smooth functions $f\colon \mathbb{R}^n \to E$ with support in the closure of V, provided with its natural topology (see [4, page 247], [12, page 450]). When $E = \mathbb{C}$ we simply write $\mathcal{S}(V)$.

Estrada [3] generalized Durán's result:

(i) first, he solved the following moment problem by constructing a solution in series form:

Lemma 1.1. ([3, Lemma 2.2]) *Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of elements of E. Then there exists a bounded, continuous and rapidly decreasing at infinity function $F\colon (0,\infty) \to E$ such that $M(f) = \{a_m \colon m \in \mathbb{N}\}$.*

(ii) secondly, from 1.1 and by the use of convolution of functions, he obtained

Proposition 1.2. ([3, Theorem 2.4]) *Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of elements of E. Then there exists $f \in \mathcal{S}((0,\infty),E)$ such that $M(f) = \{a_m \colon m \in \mathbb{N}\}$.*

(iii) lastly, the result 1.2 and the fact that the spaces

$$\mathcal{S}((0,\infty),\mathcal{S}((0,\infty)^{n-1},E)) \qquad \text{and} \qquad \mathcal{S}((0,\infty)^n,E) \qquad (n \geq 2)$$

are isomorphic allow to apply an induction argument on the number of variables in order to obtain

Theorem 1.3. ([3, Theorem 4.1]) *Let $\{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ be a family of elements of E. Then there exists $f \in \mathcal{S}((0,\infty)^n,E)$ such that*

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x} = a_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathbb{N}^n,$$

that is, $M(f) = \{a_{\boldsymbol{\alpha}} \colon \boldsymbol{\alpha} \in \mathbb{N}^n\}$.

Let us observe that, given a function f of n variables ($n \geq 2$), it is possible to integrate its products by monomials with respect to some of the variables, obtaining in this way functions that depend on the rest of them. To be precise, let us write $N = \{1, 2, \ldots, n\}$, and let J be a nonempty subset of N. We set $J^c = N \setminus J$, and for $J \subset H \subset N$, $\mathrm{proy}_{H,J}$ represents the natural projection of $\mathbb{C}^H$ into $\mathbb{C}^J$. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define $\boldsymbol{x}_J = \mathrm{proy}_{N,J}(\boldsymbol{x}) \in \mathbb{R}^J$; in particular, these notations will apply to multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^n$.

For $f \in \mathcal{S}((0,\infty)^n, E)$, $\emptyset \neq J \subset N$, $J \neq N$, and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, we define the function $f_{\boldsymbol{\alpha}_J} \colon \mathbb{R}^{J^c} \to E$ by

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{x}_{J^c}) = \int_{\mathbb{R}^J} f(\boldsymbol{x}) \boldsymbol{x}_J^{\boldsymbol{\alpha}_J} \, d\boldsymbol{x}_J, \quad \boldsymbol{x}_{J^c} \in \mathbb{R}^{J^c};$$

one has $f_{\boldsymbol{\alpha}_J} \in \mathcal{S}((0,\infty)^{J^c}, E)$. For $\boldsymbol{\alpha} \in \mathbb{N}^n \equiv \mathbb{N}^N$ (that is, when $J = N$),

$$f_{\boldsymbol{\alpha}} = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \boldsymbol{x}^{\boldsymbol{\alpha}} \, d\boldsymbol{x} \in E$$

is precisely the *moment of order* $\boldsymbol{\alpha}$ of f. Generally speaking, we call $f_{\boldsymbol{\alpha}_J}$ the *moment of order* $\boldsymbol{\alpha}_J$ of f, and (admitting that $\mathcal{S}((0,\infty)^{N^c}, E) = E$) we may associate with f the *family of generalized moments*,

$$\mathrm{GM}(f) = \{ f_{\boldsymbol{\alpha}_J} \in \mathcal{S}((0,\infty)^{J^c}, E) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \}.$$

$\mathrm{GM}(f)$ and $\mathrm{M}(f)$ coincide only for functions f of one variable.

As an easy consequence of Fubini's theorem, we find the following *compatibility conditions* relating the elements of a family of generalized moments to each other. For any pair J and L of nonempty disjoint subsets of N, for $\boldsymbol{x}_J \in \mathbb{R}^J$ and $\boldsymbol{x}_L \in \mathbb{R}^L$, $(\boldsymbol{x}_J, \boldsymbol{x}_L)$ stands for the element of $\mathbb{R}^{J \cup L}$ such that

$$\mathrm{proy}_{J\cup L, J}(\boldsymbol{x}_J, \boldsymbol{x}_L) = \boldsymbol{x}_J \quad \text{and} \quad \mathrm{proy}_{J\cup L, L}(\boldsymbol{x}_J, \boldsymbol{x}_L) = \boldsymbol{x}_L.$$

Proposition 1.4. [compatibility conditions] *Let $f \in \mathcal{S}((0,\infty)^n, E)$. Then, for every pair J and L of nonempty disjoint subsets of N, and for every $\boldsymbol{\alpha}_J \in \mathbb{N}^J$ and $\boldsymbol{\alpha}_L \in \mathbb{N}^L$, we have*

$$\int_{\mathbb{R}^L} f_{\boldsymbol{\alpha}_J}(\boldsymbol{x}_{J^c}) \boldsymbol{x}_L^{\boldsymbol{\alpha}_L} \, d\boldsymbol{x}_L = f_{(\boldsymbol{\alpha}_J, \boldsymbol{\alpha}_L)}(\boldsymbol{x}_{(J\cup L)^c}), \qquad \boldsymbol{x}_{(J\cup L)^c} \in \mathbb{R}^{(J\cup L)^c}$$

(with the obvious meaning when $J \cup L = N$).

Our main objective is to obtain an interpolation result in terms of families of generalized moments for functions of several variables:

Given a compatible (that is, under the conditions in 1.4) family

$$\mathcal{F} = \{ g_{\boldsymbol{\alpha}_J} \in \mathcal{S}((0,\infty)^{J^c}, E) \},$$

there exists $f \in \mathcal{S}((0,\infty)^n, E)$ with $\mathrm{GM}(f) = \mathcal{F}$.

The paper is organized as follows. In Section 2 we give (Theorem 2.1) an alternative construction (in series form) of the solution for Theorem 1.3, based on the same idea as that of Estrada for the proof of Lemma 1.1. This technique avoids using convolution, and allows us to deduce easily additional information on the moments of the interpolating function in the case that the data belong to a Fréchet space of the type $\mathcal{S}((0,\infty)^p, E)$ and some of its moments are predetermined as null. In this sense we provide results 2.2 and 3.1, that turn out to be essential for what follows. We close this section with an interpolation result (Theorem 2.3) which could be considered as intermediate between Theorems 2.1 (classical moment problem) and 3.2 (generalized moment problem). In fact, Theorem 2.3 solves the generalized problem in the particular case of functions of two variables. Its proof has been included since, in our opinion, it clarifies the rest of the exposition.

Section 3 is devoted to the solution of the problem initially stated. For $f \in \mathcal{S}((0,\infty)^n, E)$ the family $\mathrm{GM}(f)$ is determined by its *first order* subfamily $\mathrm{FM}(f)$, consisting of the elements of $\mathrm{GM}(f)$ that depend on $n-1$ variables, and $\mathrm{FM}(f)$ is subject to the corresponding compatibility conditions. This allows us to reformulate the problem in terms of families such as

$$\mathcal{G} = \{\, g_{m_{\{j\}}} \in \mathcal{S}((0,\infty)^{\{j\}^c}, E) : j \in N,\ m \in \mathbb{N}\,\},$$

which we call *of first order*. We prove the following result (Theorem 3.2):

Given a compatible first order family $\mathcal{G} = \{g_{m_{\{j\}}}\}$, there exists $f \in \mathcal{S}((0,\infty)^n, E)$ with $\mathrm{FM}(f) = \mathcal{G}$.

It may be worth mentioning that both the statement and the solution of this generalized moment problem have been motivated by the study (see [7, 8, 5, 6, 9, 10]) of Borel-Ritt type interpolation problems for strongly asymptotically developable functions on polysectors. In fact, the links between these two areas can shed new light on some problems (see, for example, [2]).

2 Vector moment problem revisited

In this Section we intend to solve the following problem:

Consider the families $\{f_{\boldsymbol{\beta}}\}_{\boldsymbol{\beta}\in\mathbb{N}^p}$ and $\{g_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$, whose elements belong to the Fréchet spaces $\mathcal{S}((0,\infty)^n)$ and $\mathcal{S}((0,\infty)^p)$, respectively. Find a function $f \in \mathcal{S}((0,\infty)^{n+p})$ such that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ and every $\boldsymbol{\beta} \in \mathbb{N}^p$ one has

$$\int_{\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x} = g_{\boldsymbol{\alpha}}(\boldsymbol{y}), \quad \boldsymbol{y}\in\mathbb{R}^p; \quad \int_{\mathbb{R}^p} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y} = f_{\boldsymbol{\beta}}(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^n.$$

For the problem to be solvable the data must verify certain compatibility conditions, emanated from Fubini's theorem: for every $\boldsymbol{\beta}\in\mathbb{N}^p$, $\boldsymbol{\alpha}\in\mathbb{N}^n$,

$$\int_{\mathbb{R}^n} f_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x} = \int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}.$$

As indicated in the introduction, minor modifications in the proof supplied by Estrada for Lemma 1.1 allow us to construct, in series form, a solution to the classical moment problem for vector functions of several variables. This will be crucial for what follows, so that details are supplied in the next result. Before stating it, we note that Theorem 1.3 implies the existence, for every $\boldsymbol{\alpha}\in\mathbb{N}^n$, of a function $\phi_{\boldsymbol{\alpha}} \in \mathcal{S}((0,\infty)^n)$ such that

$$\int_{\mathbb{R}^n} \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\beta}}\,d\boldsymbol{x} = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \quad \boldsymbol{\beta}\in\mathbb{N}^n.$$

For $f \in \mathcal{S}(\mathbb{R}^n)$ and $k\in\mathbb{N}$, we put

$$\|f\|_k = \sup\{|\boldsymbol{x}^{\boldsymbol{\beta}} D^{\boldsymbol{\gamma}} f(\boldsymbol{x})| : \boldsymbol{x}\in\mathbb{R}^n, 0\le|\boldsymbol{\beta}|\le k,\ 0\le|\boldsymbol{\gamma}|\le k\}.$$

Also, we write

$$M_k = \sup\{\|\phi_{\boldsymbol\beta}\|_j : 0 \le |\boldsymbol\beta| \le k,\ 0 \le j \le k\}, \qquad k \in \mathbb{N}.$$

Theorem 2.1. *Let the topology of the Fréchet space E be defined by the increasing sequence $\{p_n\}_{n=0}^{\infty}$ of seminorms. Given an arbitrary family $\{a_{\boldsymbol\alpha}\}_{\boldsymbol\alpha\in\mathbb{N}^n}$ of elements of E, choose a family $\{\lambda_{\boldsymbol\alpha}\}_{\boldsymbol\alpha\in\mathbb{N}^n}$ of real numbers, $0 < \lambda_{\boldsymbol\alpha} < 1$, such that*

$$\sum_{\boldsymbol\alpha\in\mathbb{N}^n} \lambda_{\boldsymbol\alpha}\, M_{|\boldsymbol\alpha|}\, p_{|\boldsymbol\alpha|}(a_{\boldsymbol\alpha}) < \infty.$$

Then the series

$$\sum_{\boldsymbol\alpha\in\mathbb{N}^n} \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|+n}\, \phi_{\boldsymbol\alpha}(\lambda_{\boldsymbol\alpha}\boldsymbol{x})\, a_{\boldsymbol\alpha}, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{1}$$

defines a function $f \in \mathcal{S}((0,\infty)^n, E)$ such that $M(f) = \{a_{\boldsymbol\alpha} \colon \boldsymbol\alpha \in \mathbb{N}^n\}$.

Proof. Let us observe that, given $\boldsymbol\beta, \boldsymbol\gamma \in \mathbb{N}^n$ and $m \in \mathbb{N}$, for every $\boldsymbol\alpha \in \mathbb{N}^n$ such that $|\boldsymbol\alpha| \ge \max\{|\boldsymbol\beta|, |\boldsymbol\gamma|, m\}$ we have

$$\begin{aligned} p_m(\lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|+|\boldsymbol\beta|+n} \boldsymbol{x}^{\boldsymbol\gamma} D^{\boldsymbol\beta}\phi_{\boldsymbol\alpha}(\lambda_{\boldsymbol\alpha}\boldsymbol{x}) a_{\boldsymbol\alpha}) &= \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|+|\boldsymbol\beta|-|\boldsymbol\gamma|+n} |(\lambda_{\boldsymbol\alpha}\boldsymbol{x})^{\boldsymbol\gamma} D^{\boldsymbol\beta}\phi_{\boldsymbol\alpha}(\lambda_{\boldsymbol\alpha}\boldsymbol{x})|\, p_m(a_{\boldsymbol\alpha}) \\ &\le \lambda_{\boldsymbol\alpha} M_{|\boldsymbol\alpha|}\, p_{|\boldsymbol\alpha|}(a_{\boldsymbol\alpha}), \quad \boldsymbol{x} \in \mathbb{R}^n. \end{aligned}$$

In particular, for $\boldsymbol\gamma = \mathbf{0}$ this implies that the formal partial derivatives of the series converge uniformly in $\mathbb{R}^n$, so that the series (1) defines a function f in $\mathcal{C}^\infty(\mathbb{R}^n)$ with support in $[0,\infty)^n$. Moreover, for $\boldsymbol\beta, \boldsymbol\gamma \in \mathbb{N}^n$, $m \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$ one has

$$\begin{aligned} p_m(\boldsymbol{x}^{\boldsymbol\gamma} D^{\boldsymbol\beta} f) &\le \sum_{|\boldsymbol\alpha| < \max\{|\boldsymbol\beta|,|\boldsymbol\gamma|,m\}} \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|+|\boldsymbol\beta|-|\boldsymbol\gamma|+n} \|\phi_{\boldsymbol\alpha}\|_{\max\{|\boldsymbol\beta|,|\boldsymbol\gamma|\}} p_m(a_{\boldsymbol\alpha}) \\ &+ \sum_{|\boldsymbol\alpha| \ge \max\{|\boldsymbol\beta|,|\boldsymbol\gamma|,m\}} \lambda_{\boldsymbol\alpha} M_{|\boldsymbol\alpha|} p_{|\boldsymbol\alpha|}(a_{\boldsymbol\alpha}). \end{aligned}$$

Thus we see that $f \in \mathcal{S}((0,\infty)^n, E)$.

The Lebesgue's dominated convergence theorem and the way the $\phi_{\boldsymbol\alpha}$ were chosen lead to

$$\begin{aligned} \int_{\mathbb{R}^n} f(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol\beta}\, d\boldsymbol{x} &= \sum_{\boldsymbol\alpha\in\mathbb{N}^n} \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|+n} a_{\boldsymbol\alpha} \int_{\mathbb{R}^n} \phi_{\boldsymbol\alpha}(\lambda_{\boldsymbol\alpha}\boldsymbol{x})\boldsymbol{x}^{\boldsymbol\beta}\, d\boldsymbol{x} \\ &= \sum_{\boldsymbol\alpha\in\mathbb{N}^n} \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|-|\boldsymbol\beta|} a_{\boldsymbol\alpha} \int_{\mathbb{R}^n} \phi_{\boldsymbol\alpha}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol\beta}\, d\boldsymbol{y} \\ &= \sum_{\boldsymbol\alpha\in\mathbb{N}^n} \lambda_{\boldsymbol\alpha}^{|\boldsymbol\alpha|-|\boldsymbol\beta|} a_{\boldsymbol\alpha} \delta_{\boldsymbol\alpha,\boldsymbol\beta} = a_{\boldsymbol\beta}, \quad \boldsymbol\beta \in \mathbb{N}^n, \end{aligned}$$

as desired. □

The previous construction is suitable for obtaining the following results.

Lemma 2.2. *Let $\{g_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ be a family of elements of $\mathcal{S}((0,\infty)^p)$ such that for every $\boldsymbol{\alpha}\in\mathbb{N}^n$ and $\boldsymbol{\beta}\in\mathbb{N}^p$ we have*

$$\int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}=0.$$

Then there exists a function $f\in\mathcal{S}((0,\infty)^{n+p})$ such that

$$\int_{\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=g_{\boldsymbol{\alpha}}(\boldsymbol{y}),\qquad \boldsymbol{\alpha}\in\mathbb{N}^n,\quad \boldsymbol{y}\in\mathbb{R}^p,$$

and

$$\int_{\mathbb{R}^p} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}=0,\quad \boldsymbol{\beta}\in\mathbb{N}^p.$$

Proof. Set $E=\mathcal{S}((0,\infty)^p)$ and $a_{\boldsymbol{\alpha}}=g_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha}\in\mathbb{N}^n$, and apply the preceding result to obtain the function

$$F(\boldsymbol{x})=\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}\lambda_{\boldsymbol{\alpha}}^{|\boldsymbol{\alpha}|+n}\phi_{\boldsymbol{\alpha}}(\lambda_{\boldsymbol{\alpha}}\boldsymbol{x})g_{\boldsymbol{\alpha}},\quad \boldsymbol{x}\in\mathbb{R}^n,$$

which is an element of $\mathcal{S}((0,\infty)^n,\mathcal{S}((0,\infty)^p))$ such that

$$\int_{\mathbb{R}^n} F(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=g_{\boldsymbol{\alpha}},\quad \boldsymbol{\alpha}\in\mathbb{N}^n.$$

Let us define $f:\mathbb{R}^{n+p}\to\mathbb{C}$ by

$$f(\boldsymbol{x},\boldsymbol{y})=F(\boldsymbol{x})(\boldsymbol{y}),\qquad \boldsymbol{x}\in\mathbb{R}^n,\ \boldsymbol{y}\in\mathbb{R}^p.$$

We have $f\in\mathcal{S}((0,\infty)^{n+p})$. It is easily seen that

$$\int_{\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=\int_{\mathbb{R}^n} F(\boldsymbol{x})(\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=g_{\boldsymbol{\alpha}}(\boldsymbol{y}),\quad \boldsymbol{\alpha}\in\mathbb{N}^n,$$

and for every $\boldsymbol{\beta}\in\mathbb{N}^p$ we have

$$\begin{aligned}\int_{\mathbb{R}^p} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y} &= \int_{\mathbb{R}^p}\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}\lambda_{\boldsymbol{\alpha}}^{|\boldsymbol{\alpha}|+n}\phi_{\boldsymbol{\alpha}}(\lambda_{\boldsymbol{\alpha}}\boldsymbol{x})g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}\\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}\lambda_{\boldsymbol{\alpha}}^{|\boldsymbol{\alpha}|+n}\phi_{\boldsymbol{\alpha}}(\lambda_{\boldsymbol{\alpha}}\boldsymbol{x})\int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}=0,\end{aligned}$$

which concludes the proof. □

Theorem 2.3. *Let $\{f_{\boldsymbol{\beta}}\}_{\boldsymbol{\beta}\in\mathbb{N}^p}$ and $\{g_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ be families whose elements belong to $\mathcal{S}((0,\infty)^n)$ and $\mathcal{S}((0,\infty)^p)$, respectively, and such that for every $\boldsymbol{\beta}\in\mathbb{N}^p$, $\boldsymbol{\alpha}\in\mathbb{N}^n$,*

$$\int_{\mathbb{R}^n} f_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=\int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}. \tag{2}$$

Then there exists a function $f\in\mathcal{S}((0,\infty)^{n+p})$ such that for every $\boldsymbol{\alpha}\in\mathbb{N}^n$ and every $\boldsymbol{\beta}\in\mathbb{N}^p$ one has

$$\int_{\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\,d\boldsymbol{x}=g_{\boldsymbol{\alpha}}(\boldsymbol{y}),\quad \boldsymbol{y}\in\mathbb{R}^p;\quad \int_{\mathbb{R}^p} f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\,d\boldsymbol{y}=f_{\boldsymbol{\beta}}(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathbb{R}^n.$$

Proof. By Theorem 1.3, there exists a function $G : \mathbb{R}^p \to \mathcal{S}((0,\infty)^n)$ such that

$$\int_{\mathbb{R}^p} G(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} = f_{\boldsymbol{\beta}}, \quad \boldsymbol{\beta} \in \mathbb{R}^p.$$

Define $g : \mathbb{R}^{n+p} \to \mathbb{C}$ by $g(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y})(\boldsymbol{x})$. We have $g \in \mathcal{S}((0,\infty)^{n+p})$, and

$$\int_{\mathbb{R}^p} g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\beta}\, d\boldsymbol{y} = \int_{\mathbb{R}^p} G(\boldsymbol{y})(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} = f_{\boldsymbol{\beta}}(\boldsymbol{x}), \quad \boldsymbol{\beta} \in \mathbb{N}^p,\ \boldsymbol{x} \in \mathbb{R}^n.$$

Let us consider the family $\{h_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ of elements of $\mathcal{S}((0,\infty)^p)$ defined by

$$h_{\boldsymbol{\alpha}}(\boldsymbol{y}) = g_{\boldsymbol{\alpha}}(\boldsymbol{y}) - \int_{\mathbb{R}^n} g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\, d\boldsymbol{x}, \quad \boldsymbol{\alpha} \in \mathbb{N}^n,\ \boldsymbol{y} \in \mathbb{R}^p.$$

From (2) it is clear that

$$\begin{aligned}
\int_{\mathbb{R}^p} h_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} &= \int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} - \int_{\mathbb{R}^p} \Big[\int_{\mathbb{R}^n} g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\, d\boldsymbol{x}\Big]\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y}\\
&= \int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} - \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^p} g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y}\Big]\boldsymbol{x}^{\boldsymbol{\alpha}}\, d\boldsymbol{x}\\
&= \int_{\mathbb{R}^p} g_{\boldsymbol{\alpha}}(\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} - \int_{\mathbb{R}^n} f_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}\, d\boldsymbol{x} = 0.
\end{aligned}$$

Then, the previous lemma ensures the existence of $h \in \mathcal{S}((0,\infty)^{n+p})$ such that

$$\int_{\mathbb{R}^n} h(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{\alpha}}\, d\boldsymbol{x} = h_{\boldsymbol{\alpha}}(\boldsymbol{y}), \quad \boldsymbol{\alpha} \in \mathbb{N}^n; \quad \int_{\mathbb{R}^p} h(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y}^{\boldsymbol{\beta}}\, d\boldsymbol{y} = 0, \quad \boldsymbol{\beta} \in \mathbb{N}^p.$$

The function $f = g + h \in \mathcal{S}((0,\infty)^{n+p})$ solves the problem. □

3 Generalized moment problem

We begin by reformulating the generalized moment problem. Unless otherwise stated, the natural number n will be at least 2.

For $f \in \mathcal{S}((0,\infty)^n, E)$ we may consider the *family of first order generalized moments* of f,

$$\mathrm{FM}(f) = \{\, f_{m_{\{j\}}} \in \mathcal{S}((0,\infty)^{N-\{j\}}, E) : j \in N,\ m \in \mathbb{N}\,\} \subset \mathrm{GM}(f).$$

For the sake of simplicity we will write j^c instead of $\{j\}^c = N - \{j\}$, and f_{jm} instead of $f_{m_{\{j\}}}$. We recall that

$$f_{jm}(\boldsymbol{x}_{j^c}) = \int_{\mathbb{R}^{\{j\}}} f(\boldsymbol{x}_{j^c}, x_j)x_j^m\, dx_j, \qquad \boldsymbol{x}_{j^c} \in \mathbb{R}^{j^c}.$$

Two fundamental facts can be deduced from the compatibility conditions for $\mathrm{GM}(f)$:

(i) $\mathrm{FM}(f)$ determines $\mathrm{GM}(f)$:

Indeed, let $J \subset N$ consist of at least two elements, and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$. If we choose $j \in J$, we may write

(3) $$f_{\boldsymbol{\alpha}_J}(\boldsymbol{x}_{J^c}) = \int_{\mathbb{R}^{J-\{j\}}} f_{j\alpha_j}(\boldsymbol{x}_{j^c})\boldsymbol{x}_{J-\{j\}}^{\boldsymbol{\alpha}_{J-\{j\}}}\, d\boldsymbol{x}_{J-\{j\}}, \quad \boldsymbol{x}_{J^c} \in \mathbb{R}^{J^c}.$$

(ii) $\mathrm{FM}(f)$ is subject to the corresponding *first order compatibility conditions*:
Let $j, \ell \in N$ and $\alpha_j, \alpha_\ell \in \mathbb{N}$. Then for every $\boldsymbol{x}_{\{j,\ell\}^c} \in \mathbb{R}^{\{j,\ell\}^c}$ we have

$$\int_{\mathbb{R}^{\{\ell\}}} f_{j\alpha_j}(\boldsymbol{x}_{j^c})\boldsymbol{x}_\ell^{\alpha_\ell}\, d\boldsymbol{x}_\ell = \int_{\mathbb{R}^{\{j\}}} f_{\ell\alpha_\ell}(\boldsymbol{x}_{\ell^c})\boldsymbol{x}_j^{\alpha_j}\, d\boldsymbol{x}_j$$

(with the obvious meaning when $\{j, \ell\} = N$).

Now suppose the starting point is a given family

$$\mathcal{G} = \{\, f_{jm} \in \mathcal{S}((0,\infty)^{j^c}, E) : j \in N,\ m \in \mathbb{N}\,\}$$

under the conditions in (ii) (we say $\mathcal{G}$ is a *compatible first order family*). These ensure, on the one hand, that no ambiguity appears when we use the relations in (3) in order to define, for every $J \subset N$ with at least two elements and every $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, a function $f_{\boldsymbol{\alpha}_J} \in \mathcal{S}((0,\infty)^{J^c}, E)$; and, on the other hand, that the family

$$\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \,\}$$

so defined verifies the compatibility conditions.

Summing up, there is a one-to-one and onto correspondence between the class of the families $\mathcal{F}$ and that of the families $\mathcal{G}$, subject to the respective compatibility conditions. So, the following problem is equivalent to the generalized moment problem initially stated:

Given a compatible first order family $\mathcal{G}$, find a function $f \in \mathcal{S}((0,\infty)^n, E)$ such that $\mathrm{FM}(f) = \mathcal{G}$.

In order to solve it, we begin with a lemma whose proof is analogous to the one given for Lemma 2.2.

Lemma 3.1. *Let $\{g_m\}_{m=0}^{\infty}$ be a sequence of elements of $\mathcal{S}((0,\infty)^{n-1}, E)$ such that*

$$\int_{\mathbb{R}} g_m(\boldsymbol{x}) {x_j}^q \, dx_j = 0, \quad m,\, q \in \mathbb{N},\ j \in \{1, 2, \dots, n-1\}\,.$$

Then there exists a function $f \in \mathcal{S}((0,\infty)^n, E)$ such that

$$\int_{\mathbb{R}} f(\boldsymbol{x}, y) y^m \, dy = g_m(\boldsymbol{x}), \qquad m \in \mathbb{N},$$

and

$$\int_{\mathbb{R}} f(\boldsymbol{x}, y) {x_j}^q \, dx_j = 0, \qquad q \in \mathbb{N},\ j \in \{1, 2, \dots, n-1\}\,.$$

Proof. According to the proof of Theorem 2.1, the function F given by

$$F(y) = \sum_{m=0}^{\infty} \lambda_m^{m+1} \phi_m(\lambda_m y) g_m, \qquad y \in \mathbb{R},$$

belongs to $\mathcal{S}((0,\infty), \mathcal{S}((0,\infty)^{n-1}, E))$ and

$$\int_{\mathbb{R}^n} F(y) y^m \, dy = g_m, \qquad m \in \mathbb{N}.$$

Let us define $f : \mathbb{R}^n \to E$ by

$$f(\boldsymbol{x}, y) = F(y)(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^{n-1},\ y \in \mathbb{R}.$$

We have that $f \in \mathcal{S}((0,\infty)^n, E)$ and

$$\int_{\mathbb{R}} f(\boldsymbol{x}, y) y^m \, dy = \int_{\mathbb{R}} F(y)(\boldsymbol{x}) y^m \, dy = g_m(\boldsymbol{x}), \qquad m \in \mathbb{N},\ \boldsymbol{x} \in \mathbb{R}^{n-1}.$$

Finally, for every $q \in \mathbb{N}$ and $j \in \{1, 2, \ldots, n-1\}$ we have

$$\begin{aligned} \int_{\mathbb{R}} f(\boldsymbol{x}, y) x_j^q \, dx_j &= \int_{\mathbb{R}} \sum_{m=0}^{\infty} \lambda_m^{m+1} \phi_m(\lambda_m y) g_m(\boldsymbol{x}) x_j^q \, dx_j \\ &= \sum_{m=0}^{\infty} \lambda_m^{m+1} \phi_m(\lambda_m y) \int_{\mathbb{R}} g_m(\boldsymbol{x}) x_j^q \, dx_j = 0. \end{aligned}$$

□

It is worth mentioning that the results 2.2 and 3.1 are particular cases of a more general statement which is not needed in our reasoning.

Theorem 3.2. *Let $n \geq 1$. Given a compatible first order family*

$$\mathcal{G} = \{ f_{jm} \in \mathcal{S}((0,\infty)^{j^c}, E) \colon j \in N,\ m \in \mathbb{N} \},$$

there exists a function $f \in \mathcal{S}((0,\infty)^n, E)$ such that $\mathrm{FM}(f) = \mathcal{G}$.

Proof. We apply induction on the number of variables n. For $n = 1$ the family of first order generalized moments of a function f agrees with the family of moments $M(f)$, so that the result is Theorem 1.2 of Estrada.

Suppose the result holds for $n-1$ variables, $n \geq 2$. Let $\mathcal{G} = \{ f_{jm} \in \mathcal{S}((0,\infty)^{j^c}, E) \colon j \in N,\ m \in \mathbb{N} \}$ be a compatible first order family. We recall that $\boldsymbol{x}_{n^c} = (x_1, x_2, \ldots, x_{n-1})$, $\boldsymbol{x}_{\{j,n\}^c} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1})$. For $j = 1, 2, \ldots, n-1$ let us define $\tilde{f}_{jm} \colon \mathbb{R}^{\{j,n\}^c} \to \mathcal{S}((0,\infty)^{\{n\}}, E)$ as

$$\tilde{f}_{jm}(\boldsymbol{x}_{\{j,n\}^c})(x_n) = f_{jm}(\boldsymbol{x}_{j^c}).$$

It is straightforward to check that $\tilde{f}_{jm} \in \mathcal{S}((0,\infty)^{\{j,n\}^c}, \mathcal{S}((0,\infty)^{\{n\}}, E))$, and that the family

$$\tilde{\mathcal{G}} = \{ \tilde{f}_{jm} \colon m \in \mathbb{N},\ j \in \{1, 2, \ldots, n-1\} \}$$

is a compatible first order family. By the induction hypothesis, there exists

$$\tilde{g} \in \mathcal{S}((0,\infty)^{n^c}, \mathcal{S}((0,\infty)^{\{n\}}, E))$$

such that $\mathrm{FM}(\tilde{g}) = \tilde{\mathcal{G}}$, that is,

$$\int_{\mathbb{R}^{\{j\}}} \tilde{g}(\boldsymbol{x}_{n^c}) x_j^m \, dx_j = \tilde{f}_{jm}(\boldsymbol{x}_{\{j,n\}^c}), \qquad m \in \mathbb{N},\ j \in \{1, 2, \ldots, n-1\}. \tag{4}$$

If we define $g:\mathbb{R}^n\to E$ by $g(\boldsymbol{x})=\tilde{g}(\boldsymbol{x}_{n^c})(x_n)$, we have $g\in\mathcal{S}((0,\infty)^n,E)$. Let us set $\mathrm{FM}(g)=\{g_{jm}: m\in\mathbb{N},\ j\in N\}$. From (4) and the very definition of $\tilde{f}_{jm}$ and g we get

$$\int_{\mathbb{R}\{j\}} g(\boldsymbol{x})x_j^m\,dx_j=f_{jm}(\boldsymbol{x}_{j^c}),\quad m\in\mathbb{N},\ j\in\{1,2,\ldots,n-1\},$$

or, in other words, $g_{jm}=f_{jm}$ for $m\in\mathbb{N}$ and $j\in\{1,2,\ldots,n-1\}$. Now we consider the functions $h_{nm}=f_{nm}-g_{nm}\in\mathcal{S}((0,\infty)^{n^c},E)$, $m\in\mathbb{N}$. Due to the compatibility conditions satisfied by the families $\mathcal{G}$ and $\mathrm{FM}(g)$, for every $m,q\in\mathbb{N}$ and $j\in\{1,2,\ldots,n-1\}$ we find that

$$\begin{aligned}\int_{\mathbb{R}\{j\}} h_{nm}(\boldsymbol{x}_{n^c})x_j^q\,dx_j &= \int_{\mathbb{R}\{j\}} f_{nm}(\boldsymbol{x}_{n^c})x_j^q\,dx_j-\int_{\mathbb{R}\{j\}} g_{nm}(\boldsymbol{x}_{n^c})x_j^q\,dx_j\\ &= \int_{\mathbb{R}\{n\}} f_{jq}(\boldsymbol{x}_{j^c})x_n^m\,dx_n-\int_{\mathbb{R}\{n\}} g_{jq}(\boldsymbol{x}_{j^c})x_n^m\,dx_n=0.\end{aligned}$$

By the previous lemma there exists a function $h\in\mathcal{S}((0,\infty)^n,E)$ such that

$$\int_{\mathbb{R}\{n\}} h(\boldsymbol{x})x_n^m\,dx_n=h_{nm}(\boldsymbol{x}_{n^c}),\qquad m\in\mathbb{N},$$

and

$$\int_{\mathbb{R}\{j\}} h(\boldsymbol{x})x_j^m\,dx_j=0,\qquad m\in\mathbb{N},\ j\in\{1,2,\ldots,n-1\}.$$

The function $f=g+h\in\mathcal{S}((0,\infty)^n,E)$ solves the problem. □

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