Linear continuous operators for the Stieltjes moment problem in Gelfand-Shilov spaces ⋆

Alberto Lastra [a], Javier Sanz [a,*]

[a] *Departamento de Análisis Matemático y Didáctica de la Matemática, Universidad de Valladolid, Spain*

Abstract

We obtain linear continuous operators providing a solution to the Stieltjes moment problem in the framework of Gelfand-Shilov spaces of rapidly decreasing smooth functions. The construction rests on an interpolation procedure due to R. Estrada for general rapidly decreasing smooth functions, and adapted by S.-Y. Chung, D. Kim and Y. Yeom to the case of Gelfand-Shilov spaces. It requires a linear continuous version of the so-called Borel-Ritt-Gevrey theorem in asymptotic theory.

Key words: Stieltjes moment problem, Gelfand-Shilov spaces, Gevrey asymptotics.

1 Introduction

The moment problem, initially posed and solved by T. J. Stieltjes [21], and its many generalizations have attracted the attention of mathematicians for over a century. One of these results was obtained by A. J. Durán [8]: given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, there exists a function f in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth (complex) functions,

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* Corresponding author. Address: Dpto. de Análisis Matemático y Did. de la Mat., Facultad de Ciencias, Paseo del Prado de la Magdalena s/n, 47005 Valladolid, Spain.

Email addresses: alastra@am.uva.es (Alberto Lastra), jsanzg@am.uva.es (Javier Sanz).

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with support in $[0,\infty)$ and such that

$$\int_0^\infty f(x)x^n\,dx = a_n, \qquad n \in \mathbb{N} = \{0,1,2,\ldots\}.$$

The integral on the left is the *moment of order* n of f, $\mu_n(f)$, and we will write

$$M(f) = \{\mu_n(f)\}_{n\in\mathbb{N}}. \tag{1.1}$$

Subsequently, A. L. Durán and R. Estrada [9] proved this result and some extensions by the combination of Fourier transform techniques with the Borel-Ritt theorem in asymptotic power series theory, which guarantees the existence of holomorphic functions in a given sector S of the Riemann surface of the logarithm having an arbitrarily prescribed asymptotic expansion at the vertex of S. Regarding proper subspaces of $\mathcal{S}(\mathbb{R})$, in a work of S.-Y. Chung, D. Kim and Y. Yeom [7] the moment problem is considered for the so-called Gelfand-Shilov space $\mathcal{S}_\alpha(0,\infty)$, with $\alpha > 0$, consisting of the functions $f \in \mathcal{S}(\mathbb{R})$, with support in $[0,\infty)$, for which there exists $A(f) > 0$ and, for any $\gamma \in \mathbb{N}$ there exists $C_\gamma(f) > 0$ such that

$$\sup_{x\in\mathbb{R}} |x^\beta D^\gamma f(x)| \le C_\gamma(f) A(f)^\beta \beta!^\alpha, \qquad \beta \in \mathbb{N}.$$

It is not difficult to check that whenever $f \in \mathcal{S}_\alpha(0,\infty)$, there exist $C, A > 0$ such that

$$|\mu_n(f)| \le C A^n n!^\alpha, \qquad n \in \mathbb{N}. \tag{1.2}$$

In the opposite direction, they prove the following result.

Proposition 1.1 ([7], Theorem 3.1). *Let* $\alpha > 2$*, then for every sequence* $\{\mu_n\}_{n\in\mathbb{N}}$ *satisfying bounds of type* (1.2) *there exists* $f \in \mathcal{S}_\alpha(0,\infty)$ *with* $M(f) = \{\mu_n\}_{n\in\mathbb{N}}$.

A sequence subject to bounds as in (1.2) is said to be a Gevrey sequence of order $\alpha+1$, and we write $\{\mu_n\}_{n\in\mathbb{N}} \in \Gamma_{\alpha+1}$. Our aim in this paper is to show that this result can be generalized in terms of the existence of linear continuous maps, sending a Gevrey sequence into a function, in a Gelfand-Shilov space, whose moment sequence is the one we departed from. Let us justify this approach in the following remarks.

A classical theorem of E. Borel [4] showed that the linear map

$$f \in \mathcal{C}^\infty[-1,1] \mapsto \{f^{(n)}(0)\}_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N} \tag{1.3}$$

is surjective. It is easy to see that it is also continuous when we give $\mathcal{C}^\infty[-1,1]$ its usual topology and $\mathbb{C}^\mathbb{N}$ the product topology. However, B. S. Mityagin [15] proved that there is no continuous right inverse for it, i.e., no continuous extension operator may be defined from $\mathbb{C}^\mathbb{N}$ into $\mathcal{C}^\infty[-1,1]$. So, efforts concentrated on the determination of nonquasianalytic classes of $\mathcal{C}^\infty$ functions for which this construction were possible. H.-J. Petzsche [16] characterized the classes for which the map (1.3) is surjective (onto the corresponding space of sequences), and those for which it admits a linear continuous right inverse. Similar results have been obtained regarding Whitney's extension theorem (see [3] and the references therein). As far as asymptotic theory is concerned, the situation about the aforementioned Borel-Ritt theorem is similar, as we sketch in Section 3: the continuous map sending a function, holomorphic in a sector, into its series of asymptotic expansion at the vertex admits no linear continuous right inverse (cf. Theorem 3.3). In this case, V. Thilliez [22] and, later on and with different techniques, the second author [19] proved the existence of continuous extension operators for the so-called Gevrey classes (Theorem 3.6 in this paper), so generalizing the Borel-Ritt-Gevrey theorem.

At this point we may come back to our problem. Indeed, let us observe that when we consider the usual topology in $\mathcal{S}(\mathbb{R})$ and the product topology in $\mathbb{C}^{\mathbb{N}}$, the map $M : \mathcal{S}(\mathbb{R}) \to \mathbb{C}^{\mathbb{N}}$ given in (1.1) is continuous. Also, we can easily adapt the argument of Mityagin in order to prove that there is no continuous right inverse for M (see Theorem 4.1). So, it seems natural to look for subspaces of $\mathcal{S}_\alpha(0,\infty)$ and $\Gamma_{\alpha+1}$, endowed with suitable nontrivial topologies, that allow us to construct such continuous right inverses.

The paper is organized as follows. After some notations (Section 2), we recall in Section 3 all the necessary facts about asymptotic expansions. Section 4 starts with the proof of Theorem 4.1, followed by the definition of the subspaces of $\mathcal{S}(\mathbb{R})$ that will play a crucial role in what follows, namely $\mathcal{S}_{\alpha,A}$, $\mathcal{S}^{\alpha,A}$ and $\mathcal{S}_{\alpha,A}(0,\infty)$. Next, we see the moment map M continuously sends $\mathcal{S}_{\alpha,A}(0,\infty)$ into a subspace $\Gamma_{\alpha+1,cA}$ of $\Gamma_{\alpha+1}$ for every $c > 1$ (see Proposition 4.5). After Propositions 4.6-4.8 about the way the Fourier transform takes some of these spaces into others, we state several results on the continuity of diverse operations on spaces of series, of functions with asymptotic expansion in a sector, or of smooth functions, all involved in our construction in one way or another. Finally, we reach our goal in Theorem 4.15: given $A > 0$ y $\alpha > 2$, there exists $c > 1$ and a linear continuous map $T_{\alpha,A} : \Gamma_{\alpha+1,A} \to \mathcal{S}_{\alpha,cA}(0,\infty)$ so that $M \circ T_{\alpha,A}$ is the identity map in $\Gamma_{\alpha+1,A}$. On one hand, the technique heavily rests on Theorem 3.6; indeed, the restriction on α is mainly due to the need of applying it in wide enough sectors. On the other hand, the construction procedure is essentially the same as in paper [7], where no consideration is made about continuity.

In order to conclude, we would like to mention that some of the previous results have been extended to the case of functions of several variables. In particular, the Stieltjes moment problem has been solved for rapidly decreasing smooth functions of several variables with values in a Fréchet space by R. Estrada [10], and a generalization of his result was obtained in [11]. Regarding the problem we are considering here, the extension of our main result to several variables offers no serious difficulty. The main tool needed is the corresponding version of Theorem 3.6 in several variables, which was obtained in [19] by considering the concept of strong asymptotic expansions for functions in polysectors, due to H. Majima [13,14]. Also, recent papers of J. Schmets and M. Valdivia [20] and V. Thilliez [23] provide continuous extension operators in fairly general ultraholomorphic function spaces, and we are currently preparing the corresponding version of our result for moments in generalized Gelfand-Shilov spaces.

2 Notations

We set $\mathbb{N} = \{0, 1, 2, ...\}$. Σ will denote the Riemann surface of the logarithm. A *sector* in Σ is a set

$$S = \{z : 0 < |z| < \rho,\ \alpha < \arg z < \beta\}, \qquad \rho \in (0, \infty],\ \alpha, \beta \in \mathbb{R},$$

where ρ is the radius of S. Given a sector S, we say T is a proper bounded subsector of S ($T \prec S$, for short) if it is a sector with finite radius and such that $\overline{T} - \{0\} \subseteq S$. $\mathcal{H}(S)$ denotes the set of holomorphic functions in S, and

$$\mathbb{C}[[z]] = \Big\{ \sum_{m=0}^{\infty} a_m z^m : a_m \in \mathbb{C},\ m \in \mathbb{N} \Big\}.$$

The open and closed disk with center z_0 and radius $r > 0$ will be denoted by $B(z_0, r)$ and $\overline{B}(z_0, r)$, respectively. Finally, the identity map in a set A will be written as Id_A.

3 Some facts on asymptotic expansions

For the convenience of the reader, we briefly summarize some well-known results on asymptotic expansions (an standard reference in the subject is, e.g., [1]). Let S be a sector and $f \in \mathcal{H}(S)$.

Definition 3.1. We say f admits asymptotic expansion as z tends to 0 in S, given by $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$, if for any $N \in \mathbb{N}$ and any $T \prec S$, there exists $c = c(N, T) > 0$ so that

$$\Big| f(z) - \sum_{n=0}^{N-1} f_n z^n \Big| \leq c|z|^N, \qquad z \in T.$$

In this case, one writes $f(z) \cong \hat{f}(z)$ in S.

As a consequence of Cauchy's integral formula and Taylor's formula, one easily deduces the next result.

Proposition 3.2 ([1], Proposition 8). *The following statements are equivalent:*

(i) *f admits asymptotic expansion as z tends to 0 in S.*
(ii) *For any $N \in \mathbb{N}$ and $T \prec S$ one has $\sup_{z\in T} |f^{(N)}(z)| < \infty$.*

Moreover, if $f(z) \cong \sum_{n=0}^{\infty} f_n z^n$ in S, then

$$\lim_{z\to 0, z\in T} f^{(N)}(z) = N! f_N, \qquad N \in \mathbb{N},\ T \prec S.$$

Let $\mathcal{A}(S)$ denote the set of functions $f \in \mathcal{H}(S)$ admitting an asymptotic expansion, say $\hat{f}$, at 0 in S. The map $\mathcal{J}$, defined as

$$\begin{aligned} \mathcal{J} : \mathcal{A}(S) &\longrightarrow \mathbb{C}[[z]] \\ f &\longrightarrow \hat{f}, \end{aligned} \tag{3.1}$$

is a homomorphism of differential algebras (i.e., it is linear and preservs products and derivatives; see [1, Theorems 13-15]). Let us endow the spaces $\mathcal{A}(S)$ and $\mathbb{C}[[z]]$ with natural topologies. Firstly, given $T \prec S$ and $N \in \mathbb{N}$, we put

$$p_{N,T}(f) = \sup_{z\in T} |f^{(N)}(z)|, \qquad f \in \mathcal{A}(S),$$

and we give $\mathcal{A}(S)$ the topology generated by the family $\{p_{N,T}\}_{N\in\mathbb{N}, T\prec S}$ of seminorms, which turns it into a Fréchet space. Secondly, and after the trivial identification $\mathbb{C}[[z]] \equiv \mathbb{C}^{\mathbb{N}}$, we give $\mathbb{C}^{\mathbb{N}}$ the Fréchet space structure generated by the product topology. Then, we easily see that the map $\mathcal{J}$ in (3.1) is continuous. Moreover, Borel-Ritt theorem (see [1, Theorem 16]) states that $\mathcal{J}$ is surjective for every S, so that there exists a map $T : \mathbb{C}[[z]] \longrightarrow \mathcal{A}(S)$ with $\mathcal{J} \circ T = \mathrm{Id}_{\mathbb{C}[[z]]}$. Nevertheless, we have the following

Proposition 3.3. *There does not exist a linear continuous map $T : \mathbb{C}[[z]] \to \mathcal{A}(S)$ such that $\mathcal{J} \circ T = Id_{\mathbb{C}[[z]]}$.*

PROOF. The argument is analogous to the one carried over for the proof of Proposition 4.1. □

In order to be able to construct a continuous right inverse for $\mathcal{J}$, we now consider subspaces of $\mathcal{A}(S)$ consisting of functions that admit a uniform (in a sense to be made precise) asymptotic expansion of so-called Gevrey type.

Definition 3.4. For any $\alpha \geq 1$ and $A > 0$ we define

$$\Gamma_{\alpha,A} = \Big\{ \hat{f} = \sum_{n=0}^{\infty} f_n z^n : \|\hat{f}\|_{\alpha,A} := \sup_{n\in\mathbb{N}} \frac{|f_n|}{A^n n!^{\alpha-1}} < \infty \Big\}.$$

It is easy to check that $(\Gamma_{\alpha,A}, \| \ \|_{\alpha,A})$ is a Banach space.

Definition 3.5. Let S be a sector, $\alpha \geq 1$ and $A > 0$. $\mathcal{W}_{\alpha,A}(S)$ consists of the holomorphic functions $f : S \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{W}_{\alpha,A}} := \sup_{z\in S, n\in\mathbb{N}} \frac{|f^{(n)}(z)|}{A^n n!^{\alpha}} < \infty. \tag{3.2}$$

Again, $(\mathcal{W}_{\alpha,A}(S), \| \ \|_{\mathcal{W}_{\alpha,A}})$ is a Banach space.

According to Proposition 3.2, the elements f in $\mathcal{W}_{\alpha,A}(S)$ admit asymptotic expansion at 0 in S. Indeed, since the bounds for the derivatives of f are uniform in S, rather than depending on each $T \prec S$, one can deduce that:

(i) There exist
$$f_N = \lim_{z\to 0, z\in S} \frac{f^{(N)}(z)}{N!} \in \mathbb{C}, \qquad N \in \mathbb{N},$$
and $|f_N| \leq \|f\|_{\mathcal{W}_{\alpha,A}} A^N N!^{\alpha-1}$.

(ii) f admits $\hat{f} = \sum_{n=0}^{\infty} f_n z^n$ as its asymptotic expansion of Gevrey order α uniformly in S, that is, one has
$$\Big| f(z) - \sum_{n=0}^{N-1} f_n z^n \Big| \leq \|f\|_{\mathcal{W}_{\alpha,A}} A^N N!^{\alpha-1} |z|^N, \qquad z \in S,\ N \in \mathbb{N}.$$

Hence, we can consider the map

$$\begin{aligned} \mathcal{J} : \mathcal{W}_{\alpha,A}(S) &\longrightarrow \Gamma_{\alpha,A} \\ f &\longrightarrow \mathcal{J} f = \hat{f}, \end{aligned}$$

which is linear and continuous since $\big\|\hat{f}\big\|_{\alpha,A} \leq \|f\|_{\mathcal{W}_{\alpha,A}}$. The next result [19, Theorem 4.1] provides us with the continuous right inverses for $\mathcal{J}$ in this context. Its proof is based on a refinement of Ramis' argument for the Borel-Ritt-Gevrey theorem [17,18].

Let us consider, for $\theta > 0$, the sector $S_\theta = \big\{ z \in \mathbb{C} : |\arg z + \frac{\pi}{2}| < \theta\pi/2 \big\}$.

Theorem 3.6. *Let $\alpha > 1$ and $\theta \in \mathbb{R}$ be constants such that $0 < \theta < \alpha - 1$. Then there exist $c = c(\alpha, \theta) > 1$, $C = C(\alpha, \theta) > 0$ and, for each $A \in (0, \infty)$,* *a linear map* $\Delta_{A,\theta}:\Gamma_{\alpha,A}\to\mathcal{W}_{\alpha,cA}(S_\theta)$ *such that for every* $\hat{f}\in\Gamma_{\alpha,A}$ *one has*

$$\mathcal{J}(\Delta_{A,\theta}(\hat{f}))=\hat{f},\qquad \|\Delta_{A,\theta}(\hat{f})\|_{\mathcal{W}_{\alpha,cA}}\le C\|\hat{f}\|_{\alpha,A}.$$

4 Stieltjes moment problem in Gelfand-Shilov spaces

Let us consider the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions $f:\mathbb{R}\to\mathbb{C}$, endowed with the usual structure of Fréchet space defined by the family of seminorms $(q_{\beta,\gamma})_{\beta,\gamma\in\mathbb{N}}$ given by

$$q_{\beta,\gamma}(f)=\sup_{x\in\mathbb{R}}|x^\beta D^\gamma f(x)|. \tag{4.1}$$

The space $\mathcal{S}(0,\infty)=\{f\in\mathcal{S}(\mathbb{R}):f(x)=0 \text{ for every } x\le 0\}$ is also a Fréchet space when given the subspace topology.

For every $f\in\mathcal{S}(\mathbb{R})$ one can consider its *moment sequence*, $M(f)=\{\mu_n(f)\}_{n\in\mathbb{N}}$, defined as

$$\mu_n(f)=\int_{\mathbb{R}}f(x)x^n dx,\qquad n\in\mathbb{N}.$$

When $\mathbb{C}^\mathbb{N}$ is given the product topology, the map $M:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}^\mathbb{N}$ is easily seen to be linear and continuous. Regarding its surjectivity, Durán [8] proved the following: For every $\mu=\{\mu_n\}_{n=0}^\infty\in\mathbb{C}^\mathbb{N}$ there exists $\phi\in\mathcal{S}(0,\infty)$ such that $M(\phi)=\mu$.

However, it is not possible to state this result in terms of the existence of a linear continuous right-inverse for M.

Theorem 4.1. *There does not exist a linear and continuous map* $T:\mathbb{C}^\mathbb{N}\to\mathcal{S}(0,\infty)$ *such that* $M\circ T=Id_{\mathbb{C}^\mathbb{N}}$.

PROOF. We follow the same argument in the proof of Mytiagin's result about Borel's theorem (cf. [15]). Let us suppose such T exists. Put $E_1=T(\mathbb{C}^\mathbb{N})$. Then, one easily obtains

$$E_1=\{\phi\in\mathcal{S}(0,\infty):\ \phi=TM\phi\}.$$

Hence, $E_1=(\mathrm{Id}_{\mathcal{S}(0,\infty)}-TM)^{-1}(0)$, and E_1 is closed in $\mathcal{S}(0,\infty)$. Since $MT=\mathrm{Id}_{\mathbb{C}^\mathbb{N}}$ and $TM=\mathrm{Id}_{E_1}$, we see that T and M establish an isomorphism between $\mathbb{C}^\mathbb{N}$ and E_1. This is, however, not possible, since E_1 admits a continuous norm, say

$$||\phi||=\sup_{t\in(0,\infty)}|\phi(t)|=q_{0,0}(\phi),\qquad \phi\in E_1,$$

whilst $\mathbb{C}^\mathbb{N}$ does not. □

In what follows, suitable subspaces of $\mathcal{S}(\mathbb{R})$ and $\mathbb{C}^{\mathbb{N}}$, say Σ and Γ, respectively, will be given so that, when endowed with their natural topologies, it is possible to provide linear and continuous $T : \Gamma \to \Sigma$ with $M \circ T = \mathrm{Id}_\Gamma$. To this end, we introduce the Gelfand-Shilov spaces.

Definition 4.2. Let $\alpha > 0$ and $A > 0$ be given. The space $\mathcal{S}_{\alpha,A}$ consists of those $f \in \mathcal{S}(\mathbb{R})$ such that for each $\gamma \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R},\beta\in\mathbb{N}} \frac{|x^\beta D^\gamma f(x)|}{A^\beta \beta!^\alpha} < \infty.$$

We define $\mathcal{S}_\alpha = \bigcup_{A>0} \mathcal{S}_{\alpha,A}$.

$\mathcal{S}_{\alpha,A}$ and $\mathcal{S}_\alpha$ are vector subspaces of $\mathcal{S}(\mathbb{R})$. Given $\gamma \in \mathbb{N}$, the map $p_\gamma : \mathcal{S}_{\alpha,A} \to \mathbb{R}$ defined as

$$p_\gamma(f) = \sup_{x\in\mathbb{R},\beta\in\mathbb{N}} \frac{|x^\beta D^\gamma f(x)|}{A^\beta \beta!^\alpha} \tag{4.2}$$

is a seminorm. We endow $\mathcal{S}_{\alpha,A}$ with the locally convex topology generated by the family $(p_\gamma)_{\gamma\in\mathbb{N}}$, which makes it a Fréchet space. Of course, $\mathcal{S}_{\alpha,A}(0,\infty) = \{f \in \mathcal{S}_{\alpha,A} : f(x) = 0 \text{ for every } x \in (-\infty, 0]\}$ is also a Fréchet space.

Definition 4.3. Let $\alpha > 0$ and $A > 0$ be given. The space $\mathcal{S}^{\alpha,A}$ consists of those $f \in \mathcal{S}(\mathbb{R})$ such that for each $\beta \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x^\beta D^\gamma f(x)|}{A^\gamma \gamma!^\alpha} < \infty.$$

We put $\mathcal{S}^\alpha = \bigcup_{A>0} \mathcal{S}^{\alpha,A}$.

$\mathcal{S}^{\alpha,A}$ and $\mathcal{S}^\alpha$ are vector subspaces of $\mathcal{S}(\mathbb{R})$. For every $\beta \in \mathbb{N}$, the map $p^\beta : \mathcal{S}^{\alpha,A} \to \mathbb{R}$ given by

$$p^\beta(f) = \sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x^\beta D^\gamma f(x)|}{A^\gamma \gamma!^\alpha}$$

is a seminorm, and we make $\mathcal{S}_{\alpha,A}$ a Fréchet space with the topology generated by the family $(p^\beta)_{\beta\in\mathbb{N}}$.

Remark 4.4. The elements of $\mathcal{S}_{\alpha,A}$ (resp. of $\mathcal{S}^{\alpha,A}$) admit a so-called *Gevrey type* bound with respect to the exponent of x (resp. to the order of differentiation) in $|x^\beta D^\gamma f(x)|$. All these spaces were first studied by Gelfand and Shilov [12], whom they take their name from.

The next result shows how the moment map M takes $\mathcal{S}_{\alpha,A}$ into suitable Gevrey sequence spaces.

Proposition 4.5. *Let $\alpha > 0$, $A > 0$ and $c > 1$ be given. Then, for every*

$f \in \mathcal{S}_{\alpha,A}$ one has $M(f) \in \Gamma_{\alpha+1,cA}$, and the map

$$\begin{aligned} M : \mathcal{S}_{\alpha,A}(0,\infty) &\longrightarrow \Gamma_{\alpha+1,cA} \\ f &\longrightarrow M(f) = \{\mu_n(f)\}_{n=0}^{\infty} \end{aligned}$$

is linear and continuous.

PROOF. Let $f \in \mathcal{S}_{\alpha,A}(0,\infty)$. According to (4.2), one has

$$|x^\beta f(x)| \le A^\beta \beta!^\alpha p_0(f), \qquad x > 0,\ \beta \in \mathbb{N}.$$

Hence, for every $n \in \mathbb{N}$ we deduce

$$\begin{aligned} |\mu_n(f)| &\le \int_0^\infty |f(x)x^n| dx = \int_0^1 |f(x)x^n| dx + \int_1^\infty \frac{|x^{n+2} f(x)|}{x^2} dx \\ &\le A^n n!^\alpha p_0(f) + A^{n+2}(n+2)!^\alpha p_0(f). \end{aligned} \tag{4.3}$$

There exists $C = C(\alpha) > 0$ such that

$$(n+2)^\alpha (n+1)^\alpha \le C c^n \qquad \text{for every } n \in \mathbb{N}.$$

Then, in (4.3) we get

$$|\mu_n(f)| \le (\max\{1, A\})^2 C (cA)^n n!^\alpha p_0(f), \qquad n \in \mathbb{N},$$

so that

$$\|M(f)\|_{\alpha+1,cA} := \sup_{n\in\mathbb{N}} \frac{|\mu_n(f)|}{(cA)^n n!^\alpha} \le (\max\{1, A\})^2 C p_0(f).$$

This implies that $M(f) \in \Gamma_{\alpha+1,cA}$ and M is continuous. □

For $f \in \mathcal{S}(\mathbb{R})$ we consider the Fourier transform

$$\hat{f}(w) = \mathcal{F}(f)(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt,\ w \in \mathbb{R},$$

and its inverse

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(w) e^{iwx} dw, \tag{4.4}$$

verifying that $\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{Id}_{\mathcal{S}(\mathbb{R})}$. The following result, which can be found in a more general setting in [12], shows how the Fourier transform acts on Gelfand-Shilov spaces.

Proposition 4.6. *Let $\alpha \ge 1$ and $A > 0$. Then, the following hold:*

(i) $\mathcal{F}(\mathcal{S}^{\alpha,A}) \subseteq \mathcal{S}_{\alpha,A}$, $\mathcal{F}^{-1}(\mathcal{S}^{\alpha,A}) \subseteq \mathcal{S}_{\alpha,A}$, *and the maps*

$$\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}^{\alpha,A} \longrightarrow \mathcal{S}_{\alpha,A} \tag{4.5}$$

are linear and continuous.

(ii) *For each* $c > 1$, $\mathcal{F}(\mathcal{S}_{\alpha,A}) \subseteq \mathcal{S}^{\alpha,cA}$, $\mathcal{F}^{-1}(\mathcal{S}_{\alpha,A}) \subseteq \mathcal{S}^{\alpha,cA}$, *and the maps*

$$\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}_{\alpha,A} \longrightarrow \mathcal{S}^{\alpha,cA} \tag{4.6}$$

are linear and continuous.

(iii) $\mathcal{F}(\mathcal{S}_\alpha) = \mathcal{S}^\alpha$ *and* $\mathcal{F}(\mathcal{S}^\alpha) = \mathcal{S}_\alpha$.

We recall next a characterization of the Fourier transforms of the elements in $\mathcal{S}_\alpha(0,\infty)$ (see [2]).

Proposition 4.7. *Let* ψ *be a complex function defined in* $\mathbb{R}$ *and let* $\alpha > 0$. *The following statements are equivalent:*

(i) ψ *is the Fourier transform of a function* $\phi \in \mathcal{S}_\alpha(0,\infty)$.

(ii) *The forecoming conditions hold:*

(ii.1) $\psi \in \mathcal{S}^\alpha$.

(ii.2) ψ *extends to a function* Ψ, *continuous in* $\overline{U} = \{z \in \mathbb{C} : Imz \leq 0\}$ *and analytic in* $U = \{z \in \mathbb{C} : Imz < 0\}$, *and such that* $\Psi \to 0$ *as* $z \to \infty$ *in* $\overline{U}$.

As a straightforward consequence of Propositions 4.6 and 4.7, we have

Proposition 4.8. *Let* ψ *be a complex function defined in* $\mathbb{R}$ *and let* $\alpha > 0$ *and* $A > 0$ *be given. The following statements are equivalent:*

(i') ψ *is the Fourier transform of a function* $\phi \in \mathcal{S}_{\alpha,c_1 A}(0,\infty)$ *(for certain* $c_1 \geq 1$*).*

(ii') *The conditions*

(ii'.1) $\psi \in \mathcal{S}^{\alpha,c_2 A}$ *(for certain* $c_2 \geq 1$*), and* (ii.2) *(see above)*

hold.

Our next aim is to state the continuity of several maps defined between different spaces of power series, functions having an asymptotic expansion, and smooth functions, all of them characterized by some Gevrey-like bounds.

Proposition 4.9. *Let* $\alpha > 2$ *and* $A > 0$ *be given, and let* $\sum_{n=0}^\infty a_n z^n$ *be a convergent power series with radius of convergence* $\rho \in (0,\infty]$. *Then, there exists* $c > 1$ *such that for each* $\mu = \{\mu_n\}_{n=0}^\infty \in \Gamma_{\alpha+1,A}$, *the sequence* $T_1(\mu) = \{b_n\}_{n=0}^\infty$ *defined by*

$$b_n = \sum_{k=0}^n \frac{(-i)^k}{k!} \mu_k a_{n-k}, \qquad n \in \mathbb{N}, \tag{4.7}$$

belongs to $\Gamma_{\alpha,cA}$*, and the map*

$$\begin{aligned} T_1 : \Gamma_{\alpha+1,A} &\longrightarrow \Gamma_{\alpha,cA} \\ \mu = \{\mu_n\}_{n=0}^{\infty} &\longrightarrow b = \{b_n\}_{n=0}^{\infty}, \end{aligned}$$

is linear and continuous.

PROOF. If ρ is finite, choose $c > 1$ such that $\dfrac{1}{cA} < \rho$; otherwise, c may be any real greater than 1. Recalling that

$$\|\mu\|_{\alpha+1,A} = \sup_{k\in\mathbb{N}} \frac{|\mu_k|}{A^k k!^{\alpha}},$$

for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{|b_n|}{(cA)^n n!^{\alpha-1}} &= \frac{|\sum_{k=0}^{n} \frac{(-i)^k}{k!}\mu_k a_{n-k}|}{(cA)^n n!^{\alpha-1}} \le \frac{1}{(cA)^n n!^{\alpha-1}} \sum_{k=0}^{n} \frac{|\mu_k||a_{n-k}|k!^{\alpha-1}A^k}{A^k k!^{\alpha}} \\ &\le \|\mu\|_{\alpha+1,A} \frac{1}{(cA)^n n!^{\alpha-1}} \sum_{k=0}^{n} |a_{n-k}|k!^{\alpha-1}A^k \\ &\le \frac{\|\mu\|_{\alpha+1,A}}{c^n} \sum_{k=0}^{n} \left(\frac{k!}{n!}\right)^{\alpha-1} \frac{|a_{n-k}|}{A^{n-k}} = \|\mu\|_{\alpha+1,A} \sum_{k=0}^{n} \frac{|a_{n-k}|}{(cA)^{n-k}} \frac{1}{c^k} \\ &\le \|\mu\|_{\alpha+1,A} \sum_{k=0}^{n} |a_k| \frac{1}{(cA)^k} \le \|\mu\|_{\alpha+1,A} \sum_{k=0}^{\infty} |a_k| \frac{1}{(cA)^k}. \end{aligned}$$

By virtue of the choice of c, the series $\sum_{n=0}^{\infty} a_n z^n$ absolutely converges for $z = \dfrac{1}{cA}$. Hence, $b \in \Gamma_{\alpha,cA}$, and if we put $C = \displaystyle\sum_{k=0}^{\infty} |a_k| \frac{1}{(cA)^k}$, we have

$$\|b\|_{\alpha,cA} = \sup_{n\in\mathbb{N}} \frac{|b_n|}{(cA)^n n!^{\alpha-1}} \le C\,\|\mu\|_{\alpha+1,A},$$

from where the continuity of T_1 follows. □

In the forthcoming result, the behaviour at infinity of an auxiliary function, already considered in [7], will be determined.

For $\tau > 1$ and $H = \mathbb{C} - \{iy : y \ge 1\}$ we define the function

$$h_\tau(z) = e^{\pi i(1+\frac{1}{2\tau})}(z-i)^{1/\tau}, \qquad z \in H,$$

where the determination of the logarithm is specified by $\arg(z-i) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$. Obviously, h_τ is holomorphic in H. Now, the function $G_\tau : H \to \mathbb{C}$ given by

$$G_\tau(z) = \exp(h_\tau(z)), \qquad z \in H, \tag{4.8}$$

is holomorphic in H and does not vanish, so that $\frac{1}{G_\tau}$ is analytic in H (and, in particular, in $B(0,1)$).

Lemma 4.10. *Let ε be a real number with $0 < \varepsilon < \min\left\{\frac{\pi}{2}(\tau-1), \frac{\pi}{2}\right\}$. Then, $G_\tau(z)$ tends to 0 as z tends to ∞ in the sector*

$$V_\varepsilon = \{z \in H : \arg(z-i) \in (-\pi-\varepsilon, \varepsilon)\}. \tag{4.9}$$

PROOF. For $z \in V_\varepsilon$ one has

$$\begin{aligned}\arg h_\tau(z) = \pi + \frac{\pi}{2\tau} + \frac{1}{\tau}\arg(z-i) &\in \left(\pi + \frac{\pi}{2\tau} - \frac{\pi+\varepsilon}{\tau}, \pi + \frac{\pi}{2\tau} + \frac{\varepsilon}{\tau}\right) \\ &= \left(\pi - \frac{\pi+2\varepsilon}{2\tau}, \pi + \frac{\pi+2\varepsilon}{2\tau}\right).\end{aligned}$$

Since $\dfrac{\pi+2\varepsilon}{2\tau} < \dfrac{\pi}{2}$, the image of V_ε under h_τ is contained in $\{w \in \mathbb{C} : \ \mathrm{Re} w < 0\}$; moreover, if we put

$$k_{\tau,\varepsilon} = \cos\left(\frac{\pi+2\varepsilon}{2\tau}\right) > 0,$$

one immediately sees that

$$\mathrm{Re}(h_\tau(z)) = |h_\tau(z)|\cos(\arg h_\tau(z)) \le -k_{\tau,\varepsilon}|z-i|^{\frac{1}{\tau}}, \qquad z \in V_\varepsilon.$$

So,

$$|G_\tau(z)| \le \exp(-k_{\tau,\varepsilon}|z-i|^{\frac{1}{\tau}}), \qquad z \in V_\varepsilon.$$

Since $\lim_{z\to\infty}|z-i|^{\frac{1}{\tau}} = \infty$, we are done. □

The next result deals with the product of elements in some $\mathcal{W}_{\alpha,A}(S_\theta)$ times a holomorphic and, in a sense, bounded function.

Lemma 4.11. *Let $\alpha \ge 1$, $A > 0$, $\theta > 1$ and $\varepsilon > 0$ be given, in such a way that the sector V_ε, defined in (4.9), contains the sector*

$$S_\theta = \left\{z \in \mathbb{C} : |\arg z + \frac{\pi}{2}| < \theta\pi/2\right\}.$$

Suppose G is a holomorphic function in V_ε such that for suitable $M, R > 0$ we have

$$|G(z)| \le M \qquad \text{for all } z \in V_\varepsilon \text{ with } |z| \ge R. \tag{4.10}$$

Then, there exists $c > 1$ such that for every $f \in \mathcal{W}_{\alpha,A}(S_\theta)$ one has $fG \in \mathcal{W}_{\alpha,cA}(S_\theta)$.

PROOF. Since $S_\theta \subseteq V_\varepsilon$, for each $z \in S_\theta$ one has $B(z,1/3) \subseteq V_\varepsilon$. We choose $c > 1$ such that $Ac > 3$.

For $f \in \mathcal{W}_{\alpha,A}$ one can write

$$|f^{(n)}(z)| \leq A^n n!^\alpha \|f\|_{\mathcal{W}_{\alpha,A}}, \qquad z \in S_\theta,\ n \in \mathbb{N}. \tag{4.11}$$

From

$$|(fG)^{(n)}(z)| \leq \sum_{k=0}^{n} \binom{n}{k} |f^{(k)}(z)|\ |G^{(n-k)}(z)|, \qquad z \in S_\theta,$$

and (4.11), we deduce that

$$\sup_{n\in\mathbb{N}, z\in S_\theta} \frac{|(fG)^{(n)}(z)|}{(cA)^n n!^\alpha} \leq \|f\|_{\mathcal{W}_{\alpha,A}} \sup_{n\in\mathbb{N}, z\in S_\theta} \sum_{k=0}^{n} \binom{n}{k} \frac{k!^\alpha A^k |G^{(n-k)}(z)|}{(cA)^n n!^\alpha}.$$

Let us consider

$$L = \bigcup_{z\in S_\theta} B(z, 1/3), \qquad K = \overline{L} \cap \{z \in \mathbb{C} : |z| \leq R\} \subseteq V_\varepsilon.$$

K is compact, so there exists $M_1 > 0$ with $|G(z)| \leq M_1$ for all $z \in K$. In view of (4.10) and setting $M_2 = \max\{M_1, M\}$, we see that $|G(z)| \leq M_2$ for every $z \in \overline{L}$.

For $z \in S_\theta$ we can apply Cauchy theorem in order to obtain that

$$\begin{aligned}|G^{(n-k)}(z)| &= \left| \frac{(n-k)!}{2\pi i} \int_{|w-z|=1/3} \frac{G(w)}{(w-z)^{n-k+1}} dw \right| \\ &\leq \frac{(n-k)! 2\pi \frac{1}{3}}{2\pi \left(\frac{1}{3}\right)^{n-k+1}} M_2 = \frac{(n-k)!}{\left(\frac{1}{3}\right)^{n-k}} M_2,\end{aligned}$$

and so,

$$\begin{aligned}\sup_{n\in\mathbb{N}, z\in S_\theta} \frac{|(fG)^{(n)}(z)|}{(cA)^n n!^\alpha} &\leq \|f\|_{\mathcal{W}_{\alpha,A}} M_2 \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left(\frac{k!}{n!}\right)^{\alpha-1} \frac{A^k}{\left(\frac{1}{3}\right)^{n-k} (cA)^n} \\ &\leq \|f\|_{\mathcal{W}_{\alpha,A}} M_2 \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{(cA\frac{1}{3})^{n-k}} \frac{1}{c^k} \\ &\leq \|f\|_{\mathcal{W}_{\alpha,A}} M_2 \sum_{k=0}^{\infty} \frac{1}{(cA\frac{1}{3})^k} < \infty,\end{aligned}$$

as desired. □

Remark 4.12. Given $\theta > 1$, one has $\mathbb{R} - 0 \subseteq S_\theta$. So, for a function $\Psi \in \mathcal{W}_{\alpha,A}(S_\theta)$ one can consider $\psi = \Psi|_{\mathbb{R}-\{0\}}$, which is smooth in $\mathbb{R}-\{0\}$. Moreover, it admits $\mathcal{C}^\infty$ extension to $\mathbb{R}$, since the limit at 0 of any of its derivatives exists. Indeed, if Ψ has $\sum_{n=0}^{\infty} f_n z^n$ as asymptotic expansion in S_θ, one should define $\psi^{(n)}(0) = n! f_n$, $n \in \mathbb{N}$. From now on, we will always assume that ψ is so extended.

Suppose θ and ε are suitably chosen so that $S_\theta \subseteq V_\varepsilon$. According to Lemma 4.11, the product of an element F in $\mathcal{W}_{\alpha,A}(S_\theta)$ times the function G_τ, defined in (4.8), falls into some $\mathcal{W}_{\alpha,c_0A}(S_\theta)$. By remark 4.12, $FG_\tau|_{\mathbb{R}-\{0\}}$ may be considered as a $\mathcal{C}^\infty$ function in $\mathbb{R}$. The following result gives more information about this construction.

Proposition 4.13. *Let $\tau > 1$, $\theta > 1$, $\alpha \geq 1$, $A > 0$ and $\varepsilon > 0$ be given, with*

$$S_\theta \subseteq V_\varepsilon = \{z \in H : \arg(z - i) \in (-\pi - \varepsilon, \varepsilon)\},$$

and let $G_\tau : V_\varepsilon \to \mathbb{C}$ be the function defined in (4.8)*. Then ,there exists $c > 1$ such that:*

(i) *For every $F \in \mathcal{W}_{\alpha,A}$ one has $FG_\tau|_{\mathbb{R}} \in \mathcal{S}^{\alpha,cA}$, and*
(ii) *The map*

$$\begin{aligned} T_2 : \mathcal{W}_{\alpha,A}(S_\theta) &\longrightarrow \mathcal{S}^{\alpha,cA} \\ F &\longrightarrow T_2F = (FG_\tau)|_{\mathbb{R}} \end{aligned}$$

is linear and continuous.

PROOF. For $F \in \mathcal{W}_{\alpha,A}(S_\theta)$ one has

$$\sup_{z\in S_\theta,\gamma\in\mathbb{N}} \frac{\left|F^{(\gamma)}(z)\right|}{\gamma!^\alpha A^\gamma} = \|F\|_{\mathcal{W}_{\alpha,A}} < \infty. \tag{4.12}$$

We are going to show that $T_2F \in \mathcal{S}^{\alpha,cA}$ for some $c > 1$ to be determined. Fixed $\beta \in \mathbb{N}$, we can write

$$\begin{aligned} \sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x^\beta (T_2F)^{(\gamma)}(x)|}{(cA)^\gamma \gamma!^\alpha} &= \sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x^\beta (FG_\tau)^{(\gamma)}(x)|}{(cA)^\gamma \gamma!^\alpha} \\ &\leq \sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x|^\beta \sum_{k=0}^{\gamma} \binom{\gamma}{k} |F^{(\gamma-k)}(x) G_\tau^{(k)}(x)|}{(cA)^\gamma \gamma!^\alpha}. \end{aligned} \tag{4.13}$$

It is possible to choose δ, with $0 < \delta < 1/2$, such that, whenever $x \in \mathbb{R}$ and $|x| \geq 3/2$, one has $B(x,\delta) \subseteq S_\theta$. For such an x and for every $k \in \mathbb{N}$, we have

$$|x^\beta G_\tau^{(k)}(x)| = \left| \frac{k! x^\beta}{2\pi i} \int_{|w-x|=\delta} \frac{G_\tau(w)}{(w-x)^{k+1}} dw \right| \leq \frac{k! |x|^\beta 2\pi\delta}{2\pi\delta^{k+1}} \sup_{|w-x|=\delta} |G_\tau(w)|.$$

Now, we recall that

$$G_\tau(w) = \exp\Big(\exp\Big(\pi i(1 + \frac{1}{2\tau})\Big)(w - i)^{1/\tau}\Big), \qquad w \in H.$$

For $w \in \mathbb{C}$ with $|w-x| = \delta$ it is clear that $|w-i| \geq |x| - \delta - 1$, hence

$$\left(\frac{|w-i|}{|x|}\right)^{1/\tau} \geq \left(1 - \frac{\delta+1}{|x|}\right)^{1/\tau} \geq \left(1 - \frac{\delta+1}{3/2}\right)^{1/\tau}.$$

Now, if $k_{\tau,\varepsilon}$ is the constant defined in Proposition 4.10, we have

$$-|w-i|^{1/\tau} k_{\tau,\varepsilon} \leq -|x|^{1/\tau} \left(1 - \frac{\delta+1}{3/2}\right)^{1/\tau} k_{\tau,\varepsilon}.$$

If we put $L = \left(1 - \frac{\delta+1}{3/2}\right)^{1/\tau}$, we have seen that

$$\begin{aligned}|G_\tau(w)| &= \exp(\mathrm{Re}(e^{i\pi(1+\frac{1}{2\tau})}(w-i)^{1/\tau})) \leq \exp(-k_{\tau,\varepsilon}|w-i|^{1/\tau}) \\ &\leq e^{-k_{\tau,\varepsilon}(|x|-\delta-1)^{1/\tau}} \leq \exp(-L|x|^{1/\tau} k_{\tau,\varepsilon}).\end{aligned}$$

So, setting

$$C_2 = C_2(\beta,\tau) = \max\left\{1, \max_{|x|\geq\frac{3}{2}} \left\{|x|^\beta e^{-L|x|^{1/\tau}}\right\}\right\} < \infty,$$

we can write

$$|x^\beta G_\tau^{(k)}(x)| \leq \frac{k!}{\delta^k}|x|^\beta e^{-L|x|^{1/\tau}} \leq \frac{k! C_2}{\delta^k}, \quad |x| \geq 3/2. \tag{4.14}$$

In case $|x| \leq 3/2$ we apply Cauchy's theorem: let $K \subseteq S_\theta$ be compact, and choose $\eta \in (0,1]$ such that for every x with $|x| \leq 3/2$, we have $B(x,\eta) \subseteq K \subseteq V_\varepsilon$. Put

$$M = M(\tau) = \max\left\{1, \max_{z\in K} |G_\tau(z)|\right\} < \infty.$$

Then, for any $x \in \mathbb{R}$ with $|x| \leq 3/2$, we have

$$\begin{aligned}|x^\beta G_\tau^{(k)}(x)| &\leq \left(\frac{3}{2}\right)^\beta \left|\frac{k!}{2\pi i}\int_{|w-x|=\eta} \frac{G_\tau(w)}{(w-x)^{k+1}} dw\right| \\ &\leq \left(\frac{3}{2}\right)^\beta \frac{k! 2\pi\eta}{2\pi\eta^{k+1}} \max_{z\in K}|G_\tau(z)| = \left(\frac{3}{2}\right)^\beta \frac{k!}{\eta^k} M. \end{aligned}\tag{4.15}$$

We remark that the values of δ y η depend only on θ. According to (4.14) and (4.15), we deduce that

$$|x^\beta G_\tau^{(k)}(x)| \leq \frac{(3/2)^\beta M(\tau) k! C_2(\beta,\tau)}{(\min\{\eta,\delta\})^k}, \quad x \in \mathbb{R}. \tag{4.16}$$

Let us fix $c > 1$ such that $cA\min\{\eta,\delta\} > 1$. From (4.12), (4.13) and (4.16) we see that

$$\sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{|x^\beta (T_2F)^{(\gamma)}(x)|}{(cA)^\gamma \gamma!^\alpha} \leq \sup_{x\in\mathbb{R},\gamma\in\mathbb{N}} \frac{1}{(cA)^\gamma\gamma!^\alpha} \sum_{k=0}^{\gamma} \binom{\gamma}{k} C(\tau,\beta,k)|F^{(\gamma-k)}(x)|$$

$$\leq \sup_{\gamma\in\mathbb{N}}\sum_{k=0}^{\gamma}\frac{(3/2)^{\beta}M(\tau)k!C_2(\beta,\tau)}{(\min\{\eta,\delta\})^k(cA)^{\gamma}\gamma!^{\alpha}}\binom{\gamma}{k}(\gamma-k)!^{\alpha}A^{\gamma-k}\|F\|_{\mathcal{W}_{\alpha,A}}.$$

So, if we put

$$C_1(\tau,\beta)=(3/2)^{\beta}M(\tau)C_2(\beta,\tau),$$

the last expression equals

$$C_1(\tau,\beta)\sup_{\gamma\in\mathbb{N}}\sum_{k=0}^{\gamma}\frac{k!(\gamma-k)!^{\alpha}A^{\gamma-k}\gamma!\,\|F\|_{\mathcal{W}_{\alpha,A}}}{\gamma!^{\alpha}(\min\{\eta,\delta\})^k(cA)^{\gamma}k!(\gamma-k)!}$$
$$=C_1(\tau,\beta)\sup_{\gamma\in\mathbb{N}}\sum_{k=0}^{\gamma}\left(\frac{(\gamma-k)!}{\gamma!}\right)^{\alpha-1}\frac{\|F\|_{\mathcal{W}_{\alpha,A}}}{(\min\{\eta,\delta\}cA)^k c^{\gamma-k}}$$
$$\leq C_1(\tau,\beta)\sup_{\gamma\in\mathbb{N}}\sum_{k=0}^{\gamma}\frac{\|F\|_{\mathcal{W}_{\alpha,A}}}{(\min\{\eta,\delta\}cA)^k}=C_1(\tau,\beta)\sum_{k=0}^{\infty}\frac{\|F\|_{\mathcal{W}_{\alpha,A}}}{(\min\{\eta,\delta\}cA)^k}.$$

Because of our choice of c, the previous series converges. Setting

$$C_{\beta}=C_{\tau,\beta,\theta,A}=C_1(\tau,\beta)\sum_{k=0}^{\infty}\frac{1}{(\min\{\eta,\delta\}cA)^k},$$

we conclude that

$$\sup_{x\in\mathbb{R},\gamma\in\mathbb{N}}\frac{|x^{\beta}(T_2F)^{(\gamma)}(x)|}{(cA)^{\gamma}\gamma!^{\alpha}}\leq C_{\beta}\|F\|_{\mathcal{W}_{\alpha,A}},$$

i.e., $T_2F\in\mathcal{S}^{\alpha,cA}$ and $T_2:\mathcal{W}_{\alpha,A}(S_{\theta})\longrightarrow\mathcal{S}^{\alpha,cA}$ is continuous. □

Remark 4.14. In the conditions of the previous result, the set

$$U=\{z\in\mathbb{C}:\mathrm{Im}z<0\}$$

verifies $\overline{U}-\{0\}\subseteq S_{\theta}$. So, given $F\in\mathcal{W}_{\alpha,A}(S_{\theta})$ we may consider (after the trivial extension) the function $(FG_{\tau})|_{\overline{U}}$, which is continuous in $\overline{U}$ and analytic in U, and clearly extends T_2F. Moreover, since F is bounded in S_{θ} (as is every element of $\mathcal{W}_{\alpha,A}(S_{\theta})$) and $\lim_{z\to\infty,z\in\overline{U}}G_{\tau}(z)=0$ (see Lemma 4.10), we have

$$\lim_{z\to\infty,z\in\overline{U}}(FG_{\tau})|_{\overline{U}}(z)=0.$$

In view of Proposition 4.8, T_2F is the Fourier transform of a function in $\mathcal{S}_{\alpha,cA}(0,\infty)$, with suitable c.

Finally, we prove our main result.

Theorem 4.15. *Let $A>0$ and $\alpha>2$ be given. Then, there exists $c>1$ and a linear and continuous map*

$$T_{\alpha,A}:\Gamma_{\alpha+1,A}\longrightarrow\mathcal{S}_{\alpha,cA}(0,\infty)$$

such that for every $\mu = (\mu_n)_{n=0}^{\infty} \in \Gamma_{\alpha+1,A}$ *one has* $M \circ T_{\alpha,A}(\mu) = \mu$.

PROOF. Let us fix $\tau > 1$, and let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion at 0 of the function $1/G_\tau$, analytic in $B(0,1)$. Proposition 4.9 allows us to consider a constant, say $c_1 > 1$, and the linear and continuous map $T_1 : \Gamma_{\alpha+1,A} \to \Gamma_{\alpha,c_1A}$ given, for every $\mu = \{\mu_n\}_{n=0}^{\infty} \in \Gamma_{\alpha+1,A}$, as

$$(T_1\mu)_n = \sum_{k=0}^{n} \frac{(-i)^k}{k!} \mu_k a_{n-k}, \qquad n \in \mathbb{N}.$$

Now, let us fix $\theta \in (1, \alpha-1)$. According to Theorem 3.6, there exist $c_2 > 1$ and a linear continuous map $\Delta_{c_1A,\theta} : \Gamma_{\alpha,c_1A} \to \mathcal{W}_{\alpha,c_2c_1A}(S_\theta)$ such that $\mathcal{J} \circ \Delta_{c_1A,\theta} = Id_{\Gamma_{\alpha,c_1A}}$. Next, Proposition 4.13 gives a constant $c_3 > 1$ and a linear continuous map

$$T_2 : \mathcal{W}_{\alpha,c_2c_1A}(S_\theta) \longrightarrow \mathcal{S}^{\alpha,c_3c_2c_1A}.$$

Finally, we know by Proposition 4.6 that

$$\mathcal{F}^{-1} : \mathcal{S}^{\alpha,c_3c_2c_1A} \longrightarrow \mathcal{S}_{\alpha,c_3c_2c_1A}$$

is linear and continuous. If we put $c = c_3c_2c_1 > 1$, we will show that the linear continuous map

$$T_{\alpha,A} = \mathcal{F}^{-1} \circ T_2 \circ \Delta_{c_1A,\theta} \circ T_1 : \Gamma_{\alpha+1,A} \longrightarrow \mathcal{S}_{\alpha,cA}$$

solves the problem. Indeed, according to Remark 4.14, $T_{\alpha,A}$ maps $\Gamma_{\alpha+1,A}$ into $\mathcal{S}_{\alpha,cA}(0,\infty)$. It remains to prove that $M \circ T_{\alpha,A}(\mu) = \mu$ for every $\mu \in \Gamma_{\alpha+1,A}$, or equivalently,

$$\int_0^\infty T_{\alpha,A}(\mu)(t) \cdot t^n dt = \mu_n, \qquad n \in \mathbb{N}. \tag{4.17}$$

Observe $\psi = T_2 \circ \Delta_{c_1A,\theta} \circ T_1(\mu)$ is such that $\mathcal{F} \circ T_{\alpha,A}(\mu) = \psi$, hence

$$\psi(x) = \int_0^\infty T_{\alpha,A}(\mu)(t) e^{-ixt} dt, \qquad x \in \mathbb{R},$$

and

$$\psi^{(n)}(x) = (-i)^n \int_0^\infty t^n T_{\alpha,A}(\mu)(t) e^{-ixt} dt, \qquad x \in \mathbb{R},\ n \in \mathbb{N}.$$

In particular,

$$\psi^{(n)}(0) = (-i)^n \int_0^\infty t^n T_{\alpha,A}(\mu)(t) dt, \qquad n \in \mathbb{N}.$$

Taking into account (4.17), it suffices to obtain that

$$\psi^{(n)}(0) = (-i)^n \mu_n, \qquad n \in \mathbb{N}.$$

This fact, by the very definition of T_2 and Remark 4.12, amounts to the function $\left(\Delta_{c_1A,\theta} \circ T_1(\mu)\right) \cdot G_\tau$ having

$$\sum_{n=0}^{\infty} \frac{(-i)^n \mu_n}{n!} z^n$$

as asymptotic expansion in S_θ. Now, we know the asymptotic expansion of this function is the Cauchy product of the expansions for $\Delta_{c_1A,\theta} \circ T_1(\mu)$ and G_τ, respectively. By construction, we have

$$\Delta_{c_1A,\theta} \circ T_1(\mu) \cong \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{(-i)^k}{k!} \mu_k a_{n-k} \Big) z^n \quad \text{en } S_\theta.$$

On the other hand, if $\sum_{n=0}^{\infty} a_n z^n$ is the Taylor expansion of $1/G_\tau$ at 0, the formal inverse $\sum_{n=0}^{\infty} b_n z^n$ of the former is that of G_τ. It is clear that

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{(-i)^k}{k!} \mu_k a_{n-k} \Big) z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \frac{(-i)^n \mu_n}{n!} z^n$$

(just multiply both sides by $\sum_{n=0}^{\infty} a_n z^n$), what concludes the proof. □

Remark 4.16. In the paper [6] it is proved that the function $T_{\alpha,A}(\mu)$ just constructed is also an element of the space $\mathcal{S}^\tau$, where $\tau > 1$ is the constant fixed in the proof. As can be found (in a more general setting) in [5], this implies that $T_{\alpha,A}(\mu) \in \mathcal{S}^\tau_\alpha$, i. e., there exist $A, B, C > 0$ such that

$$\sup_{x \in \mathbb{R}} |x^\beta D^\gamma T_{\alpha,A}(\mu)(x)| \leq C A^\beta B^\gamma \beta!^\alpha \gamma!^\tau, \qquad \beta \in \mathbb{N},\ \gamma \in \mathbb{N}.$$

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