

Quasi-analyticity in Carleman ultraholomorphic classes

by

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Abstract. We give a characterization for two different concepts of quasi-analyticity in Carleman ultraholomorphic classes of functions of several variables in polysectors. Also, working with strongly regular sequences, we establish generalizations of Watson's Lemma under an additional condition related to the growth index of the sequence.

1 Introduction

In 1886 H. Poincaré put forward the concept of asymptotic expansion at 0 for holomorphic functions defined in an open sector in \mathbb{C} with vertex at 0. He intended to give an analytic meaning to formal power series solutions (in general, non convergent) of ordinary differential equations at irregular singular points. With the modern formulation of his definition, the next statements turn out to be equivalent:

- (i) A function f admits asymptotic expansion in S .
- (ii) The derivatives of f remain bounded in proper and bounded subsectors of S .

Formal power series and asymptotic expansions of Gevrey type do constantly appear in the theory of algebraic ordinary differential equations and of meromorphic, linear or not, systems of ordinary differential equations at an irregular singular point. Let S be a sector with vertex at 0 in the Riemann surface of the logarithm \mathcal{R} , $\alpha \geq 1$, and f a holomorphic function in S . The following are equivalent:

- (i) f admits Gevrey asymptotic expansion of order $\alpha \geq 1$ in S (we write $f \in \mathcal{W}_\alpha(S)$).
- (ii) For every proper and bounded subsector T of S , there exist constants $c, A > 0$ such that $\sup_{z \in T} |f^{(p)}(z)| \leq cA^p p!^\alpha$, $p \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$.

It is easily deduced that the map sending $f \in \mathcal{W}_\alpha(S)$ to the sequence $(f^{(p)}(0))_{p \in \mathbb{N}_0}$, where $f^{(p)}(0) := \lim_{z \rightarrow 0, z \in T} f^{(p)}(z)$, is well defined and linear. For all these facts we refer the reader to the book of W. Balsler [1]

Generalizing this situation, given a sequence of positive real numbers $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ and a sector S with vertex at 0 in \mathcal{R} , we define $\mathcal{A}_{\mathbf{M}}(S)$ as the space of holomorphic functions f defined in S for which there exists $A > 0$ (depending on f) such that $\sup_{p \in \mathbb{N}_0, z \in S} \frac{|D^p f(z)|}{A^p p! M_p} < \infty$, and consider the Borel map \mathcal{B} sending every function $f \in \mathcal{A}_{\mathbf{M}}(S)$ to $(f^{(n)}(0))_{n \in \mathbb{N}_0}$. A classical problem is that of quasi-analyticity, that is, giving a characterization for the injectivity of \mathcal{B} . For Gevrey classes of order $\alpha > 1$ (i.e. for $M_n = n!^{\alpha-1}$, $n \in \mathbb{N}_0$), the result is classical and it is called Watson's Lemma [17]: The class is quasi-analytic if, and only if, the opening of S is greater than $\pi(\alpha - 1)$. In 1966, B. I. Korenbljum [8] solved the general problem, as we recall in Theorem 4.4, in terms of the non convergence of the logarithmic integral

$$\int^{\infty} \frac{\log T_{\widetilde{\mathbf{M}}}(r)}{r^{1+1/(\gamma+1)}} dr, \quad (1)$$

where $\widetilde{\mathbf{M}} = (n!M_n)_{n \in \mathbb{N}_0}$, $T_{\widetilde{\mathbf{M}}}$ is the Ostrowski's function associated to the sequence $\widetilde{\mathbf{M}}$ (see (3)), and the sector has opening $\gamma\pi$.

In the case of several variables, H. Majima in 1983 [10, 11] introduced the so-called strong asymptotic development. To every function f admitting strong asymptotic development in a fixed polysector, he associates a unique family $\text{TA}(f)$, named of strong asymptotic expansion of f , consisting of functions obtained, as in the one variable case, as limits of the derivatives of f when some of its variables tend to zero (see (6)). The elements of $\text{TA}(f)$ admit strong asymptotic expansion in the corresponding polysector, and are linked by certain coherence conditions (see (7)). This concept enjoys all the usual algebraic properties, and the equivalence we have firstly mentioned for the one variable case holds, as it was proved in [6] (a more accessible work is [7]). Therefore, it is natural to consider, for a given sequence of positive real numbers $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ and a polysector S in \mathcal{R}^n , the space $\mathcal{A}_{\mathbf{M}}(S)$ of the holomorphic functions f in S such that there exists $A > 0$ (depending on f) with

$$\sup_{\alpha \in \mathbb{N}_0^n, z \in S} \frac{|D^{\alpha} f(z)|}{A^{|\alpha|} |\alpha|! M_{|\alpha|}} < \infty.$$

We write $\mathfrak{F}_{\mathbf{M}}(S)$ for the set of all coherent families in S (as explained in Section 3.2) and consider the maps

$$\mathcal{B} : \mathcal{A}_{\mathbf{M}}(S) \longrightarrow \mathbb{C}^{\mathbb{N}_0^n} \quad \text{and} \quad \text{TA} : \mathcal{A}_{\mathbf{M}}(S) \longrightarrow \mathfrak{F}_{\mathbf{M}}(S),$$

where the first one is defined as $\mathcal{B}(f) := (D^\alpha f(\mathbf{0}))_{\alpha \in \mathbb{N}_0^n}$, with

$$D^\alpha f(\mathbf{0}) := \lim_{z \rightarrow \mathbf{0}, z \in S} D^\alpha f(z) \in \text{TA}(f), \quad \alpha \in \mathbb{N}_0^n.$$

These maps are homomorphisms and we say $\mathcal{A}_{\mathbf{M}}(S)$ is quasi-analytic (respectively, (s) quasi-analytic) whenever \mathcal{B} (resp. TA) is injective.

In the several variables case, the works of P. Lelong [9] and W. A. Groening [4] allowed J. A. Hernández [6] to obtain (s) quasi-analyticity results for ultraholomorphic classes in polysectors whose elements f admit estimates for their derivatives $D^\alpha f$ in terms of a multi-sequence $(M_\alpha)_{\alpha \in \mathbb{N}_0^n}$. Along the same lines, the second author [14] proved two Watson's type lemmas concerning the (s) quasi-analyticity and the quasi-analyticity of the Gevrey classes considered by Y. Haraoka [5]. Our aim in this paper is to obtain the corresponding results for the general classes $\mathcal{A}_{\mathbf{M}}(S)$ introduced above, and for both quasi-analyticity concepts.

After giving some notation, Section 3.1 is devoted to strongly regular sequences, that will appear in some of the results in last section, and to the definition of the growth index related to them. We also present the function $T_{\mathbf{M}}$. In Section 3.2 we recall the theory of strong asymptotic expansions and introduce the ultraholomorphic classes of functions we will deal with. In Section 4, Korenbljum's result will allow us to give a characterization of quasi-analyticity in several variables, in one or the other sense and for arbitrary sequences \mathbf{M} , in terms of an integral similar to (1) in which the role of γ is now played by $\bar{\gamma} = \max\{\gamma_j : j = 1, \dots, n\}$ (Proposition 4.5), or $\underline{\gamma} = \min\{\gamma_j : j = 1, \dots, n\}$ (Proposition 4.12), where $\gamma_j \pi$ stands for the opening of S_j , $j \in \{1, \dots, n\}$, with $S = \prod_{k=1}^n S_k$. Next, thanks to classical results by S. Mandelbrojt [12], we give a new sufficient condition for (s) quasi-analyticity in Proposition 4.7. When one works with strongly regular sequences \mathbf{M} , it is possible to establish generalizations of Watson's Lemma (Propositions 4.9 and 4.15) under the additional condition (15) related to the growth index. We would like to point out that these results for strongly regular sequences are new even in dimension one, and they generalize previous results by J. Schmets and M. Valdivia [15] and by V. Thilliez [16].

2 Notations

\mathbb{N} will stand for $\{1, 2, \dots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we put $\mathcal{N} = \{1, 2, \dots, n\}$. If J is a nonempty subset of \mathcal{N} , $\#J$ denotes its cardinal

number.

We will consider sectors in the Riemann surface of the logarithm \mathcal{R} with vertex at 0. Let $\theta > 0$. We will write $S_\theta = \{z : |\arg z| < \frac{\theta\pi}{2}\}$, the sector of opening $\theta\pi$ and bisecting direction $d = 0$.

Let S be a sector. A proper subsector T of S is a sector such that $\overline{T} \setminus \{0\} \subseteq S$. If moreover T is bounded, we say T is a bounded proper subsector of S , and write $T \prec S$.

A polysector is a cartesian product $S = \prod_{j=1}^n S_j \subset \mathcal{R}^n$ of sectors. A polysector T is a proper subpolysector of S if $T = \prod_{j=1}^n T_j$ with $\overline{T}_j \setminus \{0\} \subseteq S_j$, $j = 1, 2, \dots, n$. T is bounded if each one of its factors is.

Given $\mathbf{z} \in \mathcal{R}^n$, we write \mathbf{z}_J for the restriction of \mathbf{z} to J , regarding \mathbf{z} as an element of $\mathcal{R}^{\mathcal{N}}$.

Let J and L be nonempty disjoint subsets of \mathcal{N} . For $\mathbf{z}_J \in \mathcal{R}^J$ and $\mathbf{z}_L \in \mathcal{R}^L$, $(\mathbf{z}_J, \mathbf{z}_L)$ represents the element of $\mathcal{R}^{J \cup L}$ satisfying $(\mathbf{z}_J, \mathbf{z}_L)_J = \mathbf{z}_J$, $(\mathbf{z}_J, \mathbf{z}_L)_L = \mathbf{z}_L$; we also write $J' = \mathcal{N} \setminus J$, and for $j \in \mathcal{N}$ we use j' instead of $\{j\}'$. In particular, we shall use these conventions for multi-indices.

For $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in (0, \infty)^n$, we write $S_{\boldsymbol{\theta}} = \prod_{j=1}^n S_{\theta_j}$ and $S_{\boldsymbol{\theta}_J} = \prod_{j \in J} S_{\theta_j} \subset \mathcal{R}^J$.

If $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathcal{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$, we define:

$$\begin{aligned} \mathbf{1} &= (1, 1, \dots, 1), & \mathbf{e}_j &= (0, \dots, \overset{j}{1}, \dots, 0), \\ |\boldsymbol{\alpha}| &= \alpha_1 + \alpha_2 + \dots + \alpha_n, & \boldsymbol{\alpha}! &= \alpha_1! \alpha_2! \dots \alpha_n!, \\ |\mathbf{z}^\boldsymbol{\alpha}| &= |\mathbf{z}|^\boldsymbol{\alpha} = |z_1|^{\alpha_1} |z_2|^{\alpha_2} \dots |z_n|^{\alpha_n}, & \mathbf{z}^\boldsymbol{\alpha} &= z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}, \\ \mathbf{D}^\boldsymbol{\alpha} &= \frac{\partial^\boldsymbol{\alpha}}{\partial \mathbf{z}^\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_n^{\alpha_n}}, & \boldsymbol{\alpha} \leq \boldsymbol{\beta} &\Leftrightarrow \alpha_j \leq \beta_j, \quad j \in \mathcal{N}. \end{aligned}$$

For $\mathbf{J} \in \mathbb{N}_0^n$, we will frequently write $j = |\mathbf{J}|$.

3 Preliminaries

3.1 Strongly regular sequences

In what follows, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ will always stand for a sequence of positive real numbers, and we will always assume that $M_0 = 1$. We say:

- (α_0) \mathbf{M} is *logarithmically convex* if $M_n^2 \leq M_{n-1} M_{n+1}$ for every $n \in \mathbb{N}$.
- (μ) \mathbf{M} is *of moderate growth* if there exists $A > 0$ such that

$$M_{p+\ell} \leq A^{p+\ell} M_p M_\ell, \quad p, \ell \in \mathbb{N}_0. \quad (2)$$

(γ_1) \mathbf{M} satisfies the *strong non-quasianalyticity condition* if there exists $B > 0$ such that

$$\sum_{\ell \geq p} \frac{M_\ell}{(\ell + 1)M_{\ell+1}} \leq B \frac{M_p}{M_{p+1}}, \quad p \in \mathbb{N}_0.$$

\mathbf{M} is said to be *strongly regular* if it satisfies properties (α_0) , (μ) and (γ_1) .

Of course, for a strongly regular sequence \mathbf{M} the constants A and B above may be taken to be equal, and they must be not less than 1.

The measurable function $T_{\mathbf{M}} : (0, \infty) \rightarrow (0, \infty]$, which firstly appeared in this context in a work by A. Ostrowski [13], is given by

$$T_{\mathbf{M}}(r) = \sup_{p \in \mathbb{N}_0} \frac{r^p}{M_p}, \quad r > 0. \quad (3)$$

Following V. Thilliez [16], we define next the growth index of a strongly regular sequence.

Definition 3.1. Let $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence, $\gamma > 0$. We say \mathbf{M} satisfies property (P_γ) if there exist a sequence of real numbers $m' = (m'_p)_{p \in \mathbb{N}}$ and a constant $a \geq 1$ such that: (i) $a^{-1}M_p \leq M_{p-1}m'_p \leq aM_p$, $p \in \mathbb{N}$, and (ii) $((p + 1)^{-\gamma}m'_p)_{p \in \mathbb{N}}$ is increasing.

Proposition 3.2. ([16], Lemma 1.3.2). Let $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence. Then:

- (i) There exists $\gamma > 0$ such that (P_γ) is fulfilled and there also exists $a_1 > 0$ such that $a_1^p p!^\gamma \leq M_p$ for every $p \in \mathbb{N}_0$.
- (ii) There exist $\delta > 0$ and $a_2 > 0$ such that $M_p \leq a_2^p p!^\delta$ for every $p \in \mathbb{N}_0$.

Definition 3.3. Let \mathbf{M} be a strongly regular sequence. The *growth index* of \mathbf{M} is

$$\gamma(\mathbf{M}) = \sup\{\gamma \in \mathbb{R} : (P_\gamma) \text{ is fulfilled}\}.$$

According to Proposition 3.2, we have $\gamma(\mathbf{M}) \in (0, \infty)$.

Example 3.4. For the Gevrey sequence of order $\alpha > 0$, $\mathbf{M}_\alpha = (p!^\alpha)_{p \in \mathbb{N}_0}$, we have $\gamma(\mathbf{M}_\alpha) = \alpha$.

3.2 Strong asymptotic expansions and ultraholomorphic classes in polysectors

Every definition and result in this section can be generalized by considering functions with values in a complex Banach space B , sequences of elements in B , and so on. However, for our purpose it will be sufficient to consider $B := \mathbb{C}$, as it will be justified in Remark 4.2.

Let $n \in \mathbb{N}$, $n \geq 2$ and S be a polysector in \mathcal{R}^n with vertex at $\mathbf{0}$. Taking into account the conventions adopted in the list of Notations, we give the following

Definition 3.5. We say a holomorphic function $f : S \rightarrow \mathbb{C}$ admits *strong asymptotic development* in S if there exists a family

$$\mathcal{F} = \{ f_{\alpha_J} : \emptyset \neq J \subset \mathcal{N}, \alpha_J \in \mathbb{N}_0^J \},$$

where f_{α_J} is a holomorphic function defined in $S_{J'}$ whenever $J \neq \mathcal{N}$, and $f_{\alpha_J} \in \mathbb{C}$ if $J = \mathcal{N}$, in such a way that, if for every $\alpha \in \mathbb{N}_0^n$ we define the function

$$\text{App}_{\alpha}(\mathcal{F})(z) := \sum_{\emptyset \neq J \subset \mathcal{N}} (-1)^{\#J+1} \sum_{\substack{\beta_J \in \mathbb{N}_0^J \\ \beta_J \leq \alpha_J - \mathbf{1}_J}} \frac{f_{\beta_J}(z_{J'})}{\beta_J!} z_J^{\beta_J}, \quad z \in S,$$

then for every proper and bounded subpolysector T of S and every $\alpha \in \mathbb{N}_0^n$, there exists $c = c(\alpha, T) > 0$ such that

$$|f(z) - \text{App}_{\alpha}(\mathcal{F})(z)| \leq c|z|^{\alpha}, \quad z \in T. \quad (4)$$

\mathcal{F} is called the *total family* of strong asymptotic development associated to f . The map $\text{App}_{\alpha}(\mathcal{F})$, which is holomorphic in S , is the *approximant* of order α related to \mathcal{F} . We write $\mathcal{A}(S)$ for the space of holomorphic functions defined in S that admit strong asymptotic development in S .

The next result is due to J. A. Hernández [6] and it is based on a variant of Taylor's formula that appears in the work of Y. Haraoka [5].

Theorem 3.6. Let f be a holomorphic function defined in S . The following statements are equivalent:

- (i) f admits strong asymptotic development in S .

(ii) For every $\alpha \in \mathbb{N}_0^n$ and $T \prec S$, we have

$$Q_{\alpha,T}(f) := \sup_{z \in T} |D^\alpha f(z)| < \infty. \quad (5)$$

If properties (i) or (ii) are fulfilled, for every non empty subset J of \mathcal{N} and every $\alpha_J \in \mathbb{N}_0^J$ we have

$$f_{\alpha_J}(z_{J'}) = \lim_{\substack{z_J \rightarrow \mathbf{0}_J \\ z_J \in T_J}} D^{(\alpha_J, \mathbf{0}_{J'})} f(z), \quad z_{J'} \in S_{J'}, \quad (6)$$

for every $T_J \prec S_J$; the limit is uniform on every $T_{J'} \prec S_{J'}$ whenever $J \neq \mathcal{N}$, what implies that $f_{\alpha_J} \in \mathcal{A}(S_{J'})$ ($\mathcal{A}(S_{\mathcal{N}'})$ is meant to be \mathbb{C}).

Remark 3.7. According to (6), the family \mathcal{F} in Definition 3.5 is unique, and will be denoted by $\text{TA}(f)$, whereas the approximants will be $\text{App}_\alpha(f)$ from now on.

For future reference, we will make explicit the relationship between the estimates given in (4) and the ones in (5). Given T, T' polysectors, with $T \prec T' \prec S$, and $\alpha \in \mathbb{N}_0^n$, let us define

$$P_{\alpha,T}(f) := \sup_{z \in T} \frac{|f(z) - \text{App}_\alpha(f)(z)|}{|z|^\alpha},$$

then we have:

(i) $P_{\alpha,T}(f) \leq \frac{1}{\alpha!} Q_{\alpha,T}(f).$

(ii) There exists a constant $A > 0$, only depending on T and T' , such that

$$Q_{\alpha,T}(f) \leq A^{|\alpha|} \alpha! P_{\alpha,T'}(f).$$

According to the previous theorem, the space $\mathcal{A}(S)$ is stable under differentiation, what failed to hold for other concepts of asymptotic expansion in several variables (see [3, 7]).

Remark 3.8. In case $n = 1$, the concept of strong asymptotic development agrees with the usual one, and given $f \in \mathcal{A}(S)$, we have $\text{TA}(f)$ is reduced to the family of coefficients (except for the factorial numbers) of the formal power series of asymptotic expansion of f :

$$\text{TA}(f) = \{ a_m \in \mathbb{C} : m \in \mathbb{N}_0 \}, \quad \text{with} \quad f \sim \sum_{m=0}^{\infty} \frac{a_m}{m!} z^m.$$

Proposition 3.9 (Coherence conditions). Let $f \in \mathcal{A}(S)$ and

$$\text{TA}(f) = \{ f_{\alpha_J} : \emptyset \neq J \subset \mathcal{N}, \alpha_J \in \mathbb{N}_0^J \}$$

be its associated total family. Then, for every pair of nonempty disjoint subsets J and L of \mathcal{N} , every $\alpha_J \in \mathbb{N}_0^J$ and $\alpha_L \in \mathbb{N}_0^L$, and every $T_L \prec S_L$, we have

$$\lim_{\substack{z_L \rightarrow \mathbf{0} \\ z_L \in T_L}} D^{(\alpha_L, \mathbf{0}_{(J \cup L)'})} f_{\alpha_J}(z_{J'}) = f_{(\alpha_J, \alpha_L)}(z_{(J \cup L)'}); \quad (7)$$

the limit is uniform in each $T_{(J \cup L)'} \prec S_{(J \cup L)'}$ whenever $J \cup L \neq \mathcal{N}$.

From the relations (7) we immediately deduce that for every nonempty subset J of \mathcal{N} and every $\alpha_J \in \mathbb{N}_0^J$,

$$\text{TA}(f_{\alpha_J}) = \{ f_{(\alpha_J, \beta_L)} : \emptyset \neq L \subset J', \beta_L \in \mathbb{N}_0^L \}.$$

Definition 3.10. We say a family

$$\mathcal{F} = \{ f_{\alpha_J} \in \mathcal{A}(S_{J'}) : \emptyset \neq J \subset \mathcal{N}, \alpha_J \in \mathbb{N}_0^J \}$$

is *coherent* if it fulfills the conditions given in (7).

Given a polysector S , $\mathfrak{F}(S)$ will stand for the set of coherent families consisting of functions $f_{\alpha_J} \in \mathcal{A}(S_{J'})$, endowed with a vector structure in a natural way. We can consider the maps

$$\mathcal{B} : \mathcal{A}(S) \longrightarrow \mathbb{C}^{\mathbb{N}_0^n} \quad \text{and} \quad \text{TA} : \mathcal{A}(S) \longrightarrow \mathfrak{F}(S),$$

where the first one is given by

$$\mathcal{B}(f) := (D^\alpha f(\mathbf{0}))_{\alpha \in \mathbb{N}_0^n},$$

with

$$D^\alpha f(\mathbf{0}) := \lim_{\substack{z \rightarrow \mathbf{0} \\ z \in T \prec S}} D^\alpha f(z) = f_\alpha \in \text{TA}(f), \quad \alpha \in \mathbb{N}_0^n.$$

Without going into details, we mention that these maps are linear and are well behaved under differentiation.

Remark 3.11. Let $f \in \mathcal{A}(S)$. The *first order family* associated to f is given by

$$\mathcal{B}_1(f) := \{ f_{m_{\{j\}}} \in \mathcal{A}(S_{j'}) : j \in \mathcal{N}, m \in \mathbb{N}_0 \} \subset \text{TA}(f).$$

The first order family consists of the elements in the total family that depend on $n - 1$ variables. For the sake of simplicity, we will write f_{jm} instead of $f_{m_{\{j\}}}$, $j \in \mathcal{N}$, $m \in \mathbb{N}_0$. As it can be seen in [2, Section 4], by virtue of the coherence conditions, the knowledge of $\mathcal{B}_1(f)$ is enough to determine $\text{TA}(f)$ uniquely. In fact, if $\mathcal{B}_1(f)$ consists of null functions, then the same holds for $\text{TA}(f)$.

Definition 3.12. Let $n \in \mathbb{N}$ and S be a polysector in \mathcal{R}^n with vertex at $\mathbf{0}$. Given a sequence $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ and $A > 0$, we define the class $\mathcal{A}_{\mathbf{M}, A}(S)$ consisting of the holomorphic functions $f : S \rightarrow \mathbb{C}$ such that

$$\sup_{\mathbf{J} \in \mathbb{N}_0^n, \mathbf{z} \in S} \frac{|D^{\mathbf{J}} f(\mathbf{z})|}{A^{|\mathbf{J}|} M_{|\mathbf{J}|}} < \infty. \quad (8)$$

According to Theorem 3.6, every element f in $\mathcal{A}_{\mathbf{M}, A}(S)$ admits strong asymptotic development in S , in some sense “uniform”, because the limits or estimates in (4), (6) and (7) are valid in the whole corresponding polysector.

Obviously, the maps \mathcal{B} and TA can be restricted to these classes.

4 Quasi-analyticity and generalizations of Watson’s Lemma

We aim at studying the injectivity of the maps \mathcal{B} and \mathcal{B}_1 , what justifies the introduction of two different concepts of quasi-analyticity.

Definition 4.1. Let $n \in \mathbb{N}$, S be a (poly)sector in \mathcal{R}^n and $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence. We say that $\mathcal{A}_{\mathbf{M}}(S)$ is *(s) quasi-analytic* if the conditions:

- (i) $f \in \mathcal{A}_{\mathbf{M}}(S)$, and
- (ii) every element in $\text{TA}(f)$ is null (or, equivalently, every function in the family $\mathcal{B}_1(f)$ is null),

together imply that f is null in S (in other words, the class is (s) quasi-analytic if the map \mathcal{B}_1 restricted to the class is injective).

We say that $\mathcal{A}_{\mathbf{M}}(S)$ is *quasi-analytic* if the conditions:

- (i) $f \in \mathcal{A}_{\mathbf{M}}(S)$, and
- (ii) $\mathcal{B}(f)$ is the null (multi-)sequence,

together imply that f is the null function in S .

Remark 4.2. As we mentioned before, Section 3.2 can be generalized in a natural way by considering functions with values in a general Banach space B and sequences of elements in B . Let us denote by $\mathcal{A}_{\mathbf{M}}(S, B)$ the generalization of the class $\mathcal{A}_{\mathbf{M}}(S)$. If $\mathcal{A}_{\mathbf{M}}(S)$ is quasi-analytic (respectively, (s) quasi-analytic), so it is the class $\mathcal{A}_{\mathbf{M}}(S, B)$ for any complex Banach space B , as it is easily deduced from the following facts:

- (i) A map $f : S \rightarrow B$ is null if, and only if, we have $\varphi \circ f \equiv 0$ for every linear continuous functional $\varphi \in B'$.
- (ii) If $f \in \mathcal{A}_{\mathbf{M}}(S, B)$ and $\mathcal{B}(f) \equiv 0$ (resp., $\text{TA}(f) \equiv 0$), then for every linear continuous functional $\varphi \in B'$ we have $\varphi \circ f \in \mathcal{A}_{\mathbf{M}}(S)$ and $\mathcal{B}(\varphi \circ f) \equiv 0$ (resp., $\text{TA}(\varphi \circ f) \equiv 0$).

For this reason, the study of quasi-analyticity (respectively, (s) quasi-analyticity) will be carried out only for ultraholomorphic classes of complex functions.

Remark 4.3. It is worth saying that in the one variable case both definitions of quasi-analyticity coincide, because the families $\text{TA}(f)$ and $\mathcal{B}(f)$ are the same one. In the general case, quasi-analyticity implies (s) quasi-analyticity, since $\mathcal{B}(f)$ is a subfamily of $\text{TA}(f)$ for every $f \in \mathcal{A}_{\mathbf{M}}(S)$.

Notation: In this section, given a sequence $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$, we will write $\widetilde{\mathbf{M}}$ for the sequence $(p!M_p)_{p \in \mathbb{N}_0}$. $T_{\mathbf{M}}$ or $T_{\widetilde{\mathbf{M}}}$ will be the maps defined according to (3). On the other hand, we will study the convergence of integrals

$$\int_{r_0}^{\infty} g(r) dr,$$

meaning that they are considered in intervals of the form (r_0, ∞) , the value $r_0 > 0$ being irrelevant. Finally, given an element $\gamma = (\gamma_1, \dots, \gamma_n) \in$

$(0, \infty)^n$, we will write $S_\gamma = \prod_{j=1}^n S_{\gamma_j} = \{(z_1, \dots, z_n) \in \mathcal{R}^n : |\arg(z_j)| < \gamma_j \pi, j \in \mathcal{N}\}$,

$$\underline{\gamma} = \min\{\gamma_j : j = 1, \dots, n\}, \quad \bar{\gamma} = \max\{\gamma_j : j = 1, \dots, n\}.$$

The next theorem, which characterizes quasi-analytic classes in the one variable case, is due to B. I. Korenbljum [8, Theorem 3], and it is based on the classical results of S. Mandelbrojt [12]. Condition (iii) in the following theorem has been used by V. Thilliez in [16].

Theorem 4.4. Let $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence and $\gamma > 0$. The following statements are equivalent:

- (i) The class $\mathcal{A}_{\mathbf{M}}(S_\gamma)$ is quasi-analytic.
- (ii) $\int_0^\infty \frac{\log T_{\mathbf{M}}^\sim(r)}{r^{1+1/(\gamma+1)}} dr$ does not converge.

If \mathbf{M} verifies (α_0) , the following is also equivalent to the previous statements:

- (iii) $\sum_{p=0}^\infty \left(\frac{M_p}{(p+1)M_{p+1}} \right)^{1/(\gamma+1)}$ does not converge.

Thanks to this result, we can establish easy conditions equivalent to (s) quasi-analyticity in the several variables case.

Proposition 4.5. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence that fulfills (α_0) , and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$. The following statements are equivalent:

- (i) The class $\mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ is (s) quasi-analytic.
- (ii) There exists $j \in \{1, \dots, n\}$ such that $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is quasi-analytic.
- (iii) $\int_0^\infty \frac{\log T_{\mathbf{M}}^\sim(r)}{r^{1+1/(\bar{\gamma}+1)}} dr$ does not converge.
- (iv) $\sum_{p=0}^\infty \left(\frac{M_p}{(p+1)M_{p+1}} \right)^{1/(\bar{\gamma}+1)}$ does not converge.

Proof:

The equivalence between (iii) and (iv) is immediate according to Theorem 4.4. In order to demonstrate that (iii) implies (ii) it is enough to observe that $\bar{\gamma} = \gamma_j$ for some $j \in \{1, \dots, n\}$, and apply Theorem 4.4. In

the other direction, and due to the same theorem, if for some j the class $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is quasi-analytic, then

$$\int^{\infty} \frac{\log T_{\widetilde{\mathbf{M}}}(r)}{r^{1+1/(\gamma_j+1)}} dr = \infty,$$

and the same can be deduced for the integral in (iii) on observing that the integrand, positive for $r > 1$, becomes greater when we substitute the value γ_j by a greater one, $\bar{\gamma}$.

Now, we are going to proof that (iii) implies (i). We consider a function $f \in \mathcal{A}_{\mathbf{M}}(S_{\gamma})$ such that $\text{TA}(f)$ is the null family, and we will conclude f is null. Let $j \in \{1, \dots, n\}$ be an index for which $\bar{\gamma} = \gamma_j$, so that Theorem 4.4 guarantees that $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is quasi-analytic. For every $\mathbf{z}_{j'} \in S_{\gamma_{j'}}$ we define the function $\tilde{f}_{\mathbf{z}_{j'}}$ in S_{γ_j} by $\tilde{f}_{\mathbf{z}_{j'}}(z_j) = f(z_j, \mathbf{z}_{j'})$. We easily deduce that $\tilde{f}_{\mathbf{z}_{j'}} \in \mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$. Moreover, for every $m \in \mathbb{N}_0$ and every $T_j \prec S_{\gamma_j}$ we have

$$\lim_{z_j \rightarrow 0, z_j \in T_j} \tilde{f}_{\mathbf{z}_{j'}}^{(m)}(z_j) = \lim_{z_j \rightarrow 0, z_j \in T_j} D^{me_j} f(z_j, \mathbf{z}_{j'}) = f_{mj}(\mathbf{z}_{j'}) = 0.$$

So, $\mathcal{B}(f_{\mathbf{z}_{j'}})$ is the null family, and we conclude that $f_{\mathbf{z}_{j'}}$ is identically null. Varying $\mathbf{z}_{j'}$ in $S_{\gamma_{j'}}$ we conclude that f is null.

Finally, if we suppose condition (iii) is not fulfilled we are going to find a function $F \in \mathcal{A}_{\mathbf{M}}(S_{\gamma})$, not identically 0 and such that $\mathcal{B}_1(F) \equiv 0$, so that $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is not (s) quasi-analytic. Since we have

$$\int^{\infty} \frac{\log T_{\widetilde{\mathbf{M}}}(r)}{r^{1+1/(\bar{\gamma}+1)}} dr < \infty$$

and $\gamma_j \leq \bar{\gamma}$, it is obvious that

$$\int^{\infty} \frac{\log T_{\widetilde{\mathbf{M}}}(r)}{r^{1+1/(\gamma_j+1)}} dr < \infty \quad \text{for every } j \in \{1, \dots, n\}.$$

Applying Theorem 4.4, for every j we can guarantee the existence of a non-zero function $f_j \in \mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ such that $\mathcal{B}(f_j)$ is null. Let F be the map defined in S_{γ} by

$$F(\mathbf{z}) = \prod_{j=1}^n f_j(z_j), \quad \mathbf{z} = (z_1, \dots, z_n).$$

It is clear that F is holomorphic in S_γ and it is not null. Moreover, $F \in \mathcal{A}_{\mathbf{M}}(S_\gamma)$. Indeed, taking into account that \mathbf{M} fulfills property (α_0) , for every $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we have

$$\begin{aligned} |D^\alpha F(\mathbf{z})| &= |f_1^{(\alpha_1)}(z_1) \cdot \dots \cdot f_n^{(\alpha_n)}(z_n)| \\ &\leq C_1 A_1^{\alpha_1} \alpha_1! M_{\alpha_1} \cdot \dots \cdot C_n A_n^{\alpha_n} \alpha_n! M_{\alpha_n} \\ &\leq C A^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|}, \quad \mathbf{z} \in S_\gamma, \end{aligned}$$

for certain positive constants $C, C_1, \dots, C_n, A, A_1, \dots, A_n$.

Let us put $\mathcal{B}_1(F) = \{F_{jm} : j \in \mathcal{N}, m \in \mathbb{N}_0\}$. For $m \in \mathbb{N}_0, j \in \{1, \dots, n\}$ and $\mathbf{z}_{j'} \in S_{\gamma_{j'}}$ we have

$$F_{jm}(\mathbf{z}_{j'}) = \lim_{z_j \rightarrow 0, z_j \in S_{\gamma_j}} D^{me_j} F(\mathbf{z}) = \prod_{\ell=1, \ell \neq j}^n f_\ell(z_\ell) \lim_{z_j \rightarrow 0, z_j \in S_{\gamma_j}} f_j^{(m)}(z_j) = 0,$$

as desired. \square

As it can be observed, the fact that \mathbf{M} fulfills (α_0) is only used in the equivalence of (iii) and (iv), and in (i) \Rightarrow (iii).

The following auxiliary result will allow us to give a new sufficient condition for (s) quasi-analyticity.

Lemma 4.6. In the conditions of Proposition 4.5, the following statements are equivalent:

- (i) If f is a function defined and holomorphic in S_γ such that

$$|f(\mathbf{z})| \leq \frac{M_{|\boldsymbol{\alpha}|}}{|\mathbf{z}|^{|\boldsymbol{\alpha}|}}, \quad \text{for every } \mathbf{z} \in S_\gamma \text{ and } \boldsymbol{\alpha} \in \mathbb{N}_0^n, \quad (9)$$

then f is null in S_γ .

- (ii) The integral

$$\int_0^\infty \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\bar{\gamma}}} dr \quad (10)$$

does not converge.

- (iii) The series $\sum_{p=0}^\infty \left(\frac{M_p}{M_{p+1}}\right)^{1/\bar{\gamma}}$ does not converge.

Proof:

The equivalence between (ii) and (iii), whenever \mathbf{M} fulfills (α_0) , can be found in [12, Theorem 2.4.III].

(i) \Rightarrow (ii) Let us suppose that the integral in (10) converges. For every index $j \in \{1, \dots, n\}$ we then have that the integral obtained substituting $\bar{\gamma}$ by γ_j in (10) is also convergent. Applying again [12, Theorem 2.4.III], for every j we can guarantee the existence of a non-zero function f_j , holomorphic in $S_1 = \{z : |\arg(z)| < \pi/2\}$ and such that

$$|f_j(z)| \leq M_p/|z|^{\gamma_j p} \quad \text{for every } p \in \mathbb{N}_0, z \in S_1.$$

We define the function F by

$$F(\mathbf{z}) = f_1(z_1^{1/\gamma_1}) \cdot \dots \cdot f_n(z_n^{1/\gamma_n}), \quad \mathbf{z} = (z_1, \dots, z_n) \in S_\gamma.$$

It is clear that F is well defined, it is not null and it is holomorphic in S_γ . Moreover, taking into account property (α_0) , for every $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\mathbf{z} = (z_1, \dots, z_n) \in S_\gamma$ we have

$$|F(\mathbf{z})| \leq \frac{M_{\alpha_1}}{|z_1|^{\alpha_1}} \cdot \dots \cdot \frac{M_{\alpha_n}}{|z_n|^{\alpha_n}} \leq \frac{M_{|\boldsymbol{\alpha}|}}{|\mathbf{z}|^{|\boldsymbol{\alpha}|}},$$

concluding that (i) does not hold.

(ii) \Rightarrow (i) Let us consider a function f holomorphic in S_γ and such that

$$|f(\mathbf{z})| \leq M_{|\boldsymbol{\alpha}|}/|\mathbf{z}|^{|\boldsymbol{\alpha}|}, \quad \text{for every } \mathbf{z} \in S_\gamma, \boldsymbol{\alpha} \in \mathbb{N}_0^n.$$

We can suppose, without loss of generality, that $\bar{\gamma} = \gamma_n$.

For every $(z_1, \dots, z_{n-1}) \in S_{\gamma_{n'}}$, let g be the map defined in S_1 by

$$g(z_n) = f(z_1, \dots, z_{n-1}, z_n^{\gamma_n}),$$

which is a holomorphic map in S_1 . For every $p \in \mathbb{N}_0$ we can apply (9) with $\boldsymbol{\alpha} = (0, \dots, 0, p)$, obtaining that

$$|g(z_n)| = |f(z_1, \dots, z_{n-1}, z_n^{\gamma_n})| \leq \frac{M_p}{|z_n|^{\gamma_n p}}.$$

Applying again [12, Theorem 2.4.III], we deduce that g is the null function. As $(z_1, \dots, z_{n-1}) \in S_{\gamma_{n'}}$ was an arbitrary point, we conclude f identically vanishes in S_γ , as desired. \square

Proposition 4.7. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$. Then, we have:

- (i) If $\int^\infty \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\bar{\gamma}}} dr$ does not converge, $\mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ is (s) quasi-analytic.
- (ii) If \mathbf{M} verifies (α_0) and $\mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ is (s) quasi-analytic, then for every $\tilde{\gamma} > \bar{\gamma}$ we have $\sum_{p=0}^{\infty} \left(\frac{M_p}{M_{p+1}}\right)^{1/\tilde{\gamma}}$ and $\int^\infty \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\tilde{\gamma}}} dr$ do not converge.

Proof:

In both implications we will use the fact that, whenever $z \in S_{\boldsymbol{\gamma}}$ for some $\gamma > 0$, then also $1/z \in S_{\boldsymbol{\gamma}}$.

(i) Let us consider $f \in \mathcal{A}_{\mathbf{M}}(S_{\boldsymbol{\gamma}})$ with $\text{TA}(f) \equiv 0$, so that all of its approximants are null. There exist constants $C, A > 0$ such that

$$|D^{\boldsymbol{\alpha}} f(\mathbf{z})| \leq CA^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|! M_{|\boldsymbol{\alpha}|}, \quad \boldsymbol{\alpha} \in \mathbb{N}_0^n, \mathbf{z} \in S_{\boldsymbol{\gamma}}.$$

Taking into account Remark 3.7, we deduce that

$$|f(\mathbf{z})| = |f(\mathbf{z}) - \text{App}_{\boldsymbol{\alpha}}(f)(\mathbf{z})| \leq CA^{|\boldsymbol{\alpha}|} \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!} M_{|\boldsymbol{\alpha}|} |\mathbf{z}|^{\boldsymbol{\alpha}}, \quad \mathbf{z} \in S_{\boldsymbol{\gamma}}, \boldsymbol{\alpha} \in \mathbb{N}_0^n. \quad (11)$$

If $\mathbf{N} = (N_p)_{p \in \mathbb{N}_0}$ is the sequence given by $N_p = C(nA)^p M_p$ for every $p \in \mathbb{N}_0$, then from (11) and from the inequality $|\boldsymbol{\alpha}|! / \boldsymbol{\alpha}! \leq n^{|\boldsymbol{\alpha}|}$, valid for all $\boldsymbol{\alpha} \in \mathbb{N}_0^n$, we have that

$$\left| f\left(\frac{1}{\mathbf{z}}\right) \right| = \left| f\left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right) \right| \leq \frac{N_{|\boldsymbol{\alpha}|}}{|\mathbf{z}|^{\boldsymbol{\alpha}}}.$$

It is easy to check that the integral in (10) and

$$\int^\infty \frac{\log T_{\mathbf{N}}(r)}{r^{1+1/\tilde{\gamma}}} dr$$

are simultaneously convergent or divergent. By our hypothesis, we deduce the second one does not converge. Applying Lemma 4.6 we get that the map $f(1/\mathbf{z})$ is null in $S_{\boldsymbol{\gamma}}$, as desired.

(ii) Now, suppose there exists $\tilde{\gamma} > \bar{\gamma}$ such that the integral $\int^\infty \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\tilde{\gamma}}} dr$ is convergent. Lemma 4.6 tells us there is a holomorphic map f in $S_{\tilde{\boldsymbol{\gamma}}}$, not identically 0 and such that

$$|f(z)| \leq M_p / |z|^p \quad \text{for every } z \in S_{\tilde{\boldsymbol{\gamma}}}, p \in \mathbb{N}_0.$$

Now, if we define the map F in $S_{(\tilde{\gamma}, \dots, \tilde{\gamma})}$ as

$$F(\mathbf{z}) = f(1/z_1) \cdot \dots \cdot f(1/z_n), \quad \mathbf{z} = (z_1, \dots, z_n) \in S_{(\tilde{\gamma}, \dots, \tilde{\gamma})},$$

then, due to property (α_0) , we have

$$|F(\mathbf{z})| \leq M_{|\alpha|} |\mathbf{z}|^\alpha, \quad \mathbf{z} \in S_{(\tilde{\gamma}, \dots, \tilde{\gamma})}, \quad \alpha \in \mathbb{N}_0^n,$$

that is, F admits strong asymptotic development in $S_{(\tilde{\gamma}, \dots, \tilde{\gamma})}$ and all the elements of $\text{TA}(F)$ are null. According to Remark 3.7, and taking into account that S_γ is a proper subpolysector of $S_{(\tilde{\gamma}, \dots, \tilde{\gamma})}$, we deduce that $G := F|_{S_\gamma}$ belongs to $\mathcal{A}_{\mathbf{M}}(S_\gamma)$, it is not identically 0 and $\text{TA}(G)$ is the null family, so $\mathcal{A}_{\mathbf{M}}(S_\gamma)$ is not (s) quasi-analytic. \square

All the previous results are given in terms of a sequence \mathbf{M} , subject, at most, to the condition (α_0) . The following ones deal with the case of a strongly regular sequence \mathbf{M} . First of all, we will prove a result that, in the one variable setting, was already obtained by V. Thilliez [16]. Our proof is different from his and it also deals with the case of several variables.

Proposition 4.8. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence, and $\gamma = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$ such that $0 < \bar{\gamma} < \gamma(\mathbf{M})$. Then, the class $\mathcal{A}_{\mathbf{M}}(S_\gamma)$ is not (s) quasi-analytic.

Proof:

Let us take two real numbers $\tilde{\gamma}$ and η such that $\bar{\gamma} < \tilde{\gamma} < \eta < \gamma(\mathbf{M})$. We then have that property (P_η) is fulfilled, so, by Definition 3.1, there exist a sequence $(m'_p)_{p \in \mathbb{N}}$ and a constant $a \geq 1$ such that the sequence $((p+1)^{-\eta} m'_p)_{p \in \mathbb{N}}$ is increasing and $a^{-1} m_p \leq m'_p \leq a m_p$ for every $p \in \mathbb{N}$, where $m_p = M_p/M_{p-1}$. Therefore, the series

$$\sum_{p=1}^{\infty} \left(\frac{1}{m_p} \right)^{1/\tilde{\gamma}} \tag{12}$$

is of the same nature, convergent or not, as

$$\sum_{p=1}^{\infty} \left(\frac{1}{m'_p} \right)^{1/\tilde{\gamma}} = \sum_{p=1}^{\infty} \frac{1}{(p+1)^{\eta/\tilde{\gamma}}} \frac{1}{(m'_p (p+1)^{-\eta})^{1/\tilde{\gamma}}} \tag{13}$$

$$\leq \frac{1}{(m'_1 2^{-\eta})^{1/\tilde{\gamma}}} \sum_{p=1}^{\infty} \frac{1}{(p+1)^{\eta/\tilde{\gamma}}}. \tag{14}$$

Since $\tilde{\gamma} < \eta$, this last series is convergent and also the series in (12) is. It is enough to apply the second item in the previous result to conclude. \square

We now show that, under an additional hypothesis, the result in the previous proposition turns out to be an equivalence. In this way, as it will be discussed in a later example, the classical Watson's Lemma, valid for Gevrey classes, is extended.

Proposition 4.9 (Generalization of Watson's Lemma). Let \mathbf{M} be strongly regular and let us suppose that

$$\sum_{n=0}^{\infty} \left(\frac{M_n}{M_{n+1}} \right)^{1/\gamma(\mathbf{M})} = \infty \quad (15)$$

(or, in other words, $\int^{\infty} \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\gamma(\mathbf{M})}} dr = \infty$). Let $n \in \mathbb{N}$ and $\gamma \in (0, \infty)^n$. The following statements are equivalent:

- (i) $\bar{\gamma} \geq \gamma(\mathbf{M})$.
- (ii) The class $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is (s) quasi-analytic.

Proof:

We only have to proof that (i) implies (ii). If (15) is fulfilled and $\bar{\gamma} \geq \gamma(\mathbf{M})$, then $\int^{\infty} \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\bar{\gamma}}} dr$ does not converge, and it is enough to apply (i) in Proposition 4.7. \square

Example 4.10. For $\alpha > 0$ and $\beta \geq 0$ we consider the sequence $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ given by

$$M_p = p!^{\alpha} \left(\prod_{k=0}^p \log(e+k) \right)^{\beta}, \quad p \in \mathbb{N}_0.$$

It is not hard to check that \mathbf{M} is strongly regular and $\gamma(\mathbf{M}) = \alpha$. Moreover, \mathbf{M} fulfills condition (15) if, and only if, $\beta \leq \alpha$. It is important to observe that when $\beta = 0$ we get the Gevrey sequences, $\mathbf{M}_{\alpha} = (p!^{\alpha})_{p \in \mathbb{N}_0}$, and consequently for every $\alpha > 0$ we have that \mathbf{M}_{α} fulfills (15). So, the previous result generalizes Watson's Lemma.

Remark 4.11. It is an open problem to decide whether the condition $\bar{\gamma} \geq \gamma(\mathbf{M})$ implies $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is (s) quasi-analytic without the additional assumption (15).

We now study the quasi-analyticity of the class $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$.

Proposition 4.12. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence, and $\gamma = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$. The following statements are equivalent:

- (i) The class $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is quasi-analytic.
- (ii) For every $j \in \{1, \dots, n\}$ the class $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is quasi-analytic.
- (iii) $\int_0^{\infty} \frac{\log T_{\tilde{\mathbf{M}}}(r)}{r^{1+1/(\underline{\gamma}+1)}} dr$ does not converge.

If \mathbf{M} fulfills (α_0) , the following is also equivalent to the previous statements:

- (iv) $\sum_{p=0}^{\infty} \left(\frac{M_p}{(p+1)M_{p+1}} \right)^{1/(\underline{\gamma}+1)}$ does not converge.

Proof:

The equivalence between (iii) and (iv), when (α_0) is fulfilled, has already been discussed. In order to guarantee the equivalence between (ii) and (iii) we only have to make use of Theorem 4.4, and take into account that $\underline{\gamma} \leq \gamma_j$ for every $j \in \mathcal{N}$, that there exists j for which the equality holds, and that, when we replace the value of $\underline{\gamma}$ by a greater one, the integrand increases.

We now prove that (i) implies (ii). Let us suppose there exists $j \in \mathcal{N}$ such that $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is not quasi-analytic. Consider a non zero map $f_j \in \mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ such that $\mathcal{B}(f_j)$ is null. The map f defined in S_{γ} by $f(\mathbf{z}) = f_j(z_j)$, for $\mathbf{z} = (z_1, \dots, z_n)$, is clearly an element of $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$, it is not null, and $\mathcal{B}(f) = (0)_{\alpha \in \mathbb{N}_0^n}$: indeed, if α is such that $\alpha_{j'} = \mathbf{0}_{j'}$, then

$$\lim_{\mathbf{z} \rightarrow \mathbf{0}, \mathbf{z} \in S_{\gamma}} D^{\alpha} f(\mathbf{z}) = \lim_{z_j \rightarrow 0, z \in S_{\gamma_j}} f_j^{(\alpha_j)}(z_j) = 0,$$

while if $\alpha_{j'} \neq \mathbf{0}_{j'}$, then

$$\lim_{\mathbf{z} \rightarrow \mathbf{0}, \mathbf{z} \in S_{\gamma}} D^{\alpha} f(\mathbf{z}) = \lim_{\mathbf{z} \rightarrow \mathbf{0}, \mathbf{z} \in S_{\gamma}} 0 = 0.$$

We then deduce that $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is not quasi-analytic.

Finally, we will prove (ii) \Rightarrow (i). Let $n \in \mathbb{N}$, $n > 1$, and let $f \in \mathcal{A}_{\mathbf{M}}(S_{\gamma})$ be such that $\mathcal{B}(f) = (0)_{\alpha \in \mathbb{N}_0^n}$. For every $\alpha_{1'} \in \mathbb{N}_0^{1'}$, the map $f_{\alpha_{1'}} \in \text{TA}(f)$

belongs to $\mathcal{A}_{\mathbf{M}}(S_{\gamma_1})$ and also $\mathcal{B}(f_{\alpha_{1'}}) = (0)_{m \in \mathbb{N}_0}$, due to the coherence conditions given in (7), since for every $m \in \mathbb{N}_0$ we have

$$\lim_{z_1 \rightarrow 0, z_1 \in S_{\gamma_1}} f_{\alpha_{1'}}^{(m)}(z_1) = \lim_{z \rightarrow 0, z \in S_{\gamma}} D^{me_1} f(z) = 0.$$

By virtue of (ii) we have $\mathcal{A}_{\mathbf{M}}(S_{\gamma_1})$ is quasi-analytic, so $f_{\alpha_{1'}}$ is null in S_{γ_1} for every $\alpha_{1'} \in \mathbb{N}_0^{1'}$. Let us fix $z_1 \in S_{\gamma_1}$. For every $\alpha_{\{1,2\}'} \in \mathbb{N}_0^{\{1,2\}'}$ we consider the map $f_{\alpha_{\{1,2\}'}}(z_1, \cdot) : S_{\gamma_2} \rightarrow \mathbb{C}$. As we have $f_{\alpha_{\{1,2\}'}} \in \mathcal{A}_{\mathbf{M}}(S_{\gamma_{\{1,2\}}})$, it is clear that $f_{\alpha_{\{1,2\}'}}(z_1, \cdot) \in \mathcal{A}_{\mathbf{M}}(S_{\gamma_2})$. Moreover, due to the coherence conditions,

$$\lim_{z_2 \rightarrow 0, z_2 \in S_{\gamma_2}} D^{(0,m)} f_{\alpha_{\{1,2\}'}}(z_1, z_2) = f_{(m_{\{2\}'}, \alpha_{\{1,2\}'})}(z_1) = 0, \quad \text{for every } m \in \mathbb{N}_0.$$

Therefore, $\mathcal{B}(f_{\alpha_{\{1,2\}'}}(z_1, \cdot)) = (0)_{m \in \mathbb{N}_0}$, and as $\mathcal{A}_{\mathbf{M}}(S_{\gamma_2})$ is quasi-analytic, we have $f_{\alpha_{\{1,2\}'}}(z_1, \cdot)$ is null in S_{γ_2} . Varying z_1 in S_{γ_1} we conclude $f_{\alpha_{\{1,2\}'}} \equiv 0$ in $S_{\gamma_{\{1,2\}'}}$ for every $\alpha_{\{1,2\}'} \in \mathbb{N}_0^{\{1,2\}'}$.

In the next step, only necessary if $n > 2$, we fix $\mathbf{z}_{\{1,2\}} \in S_{\gamma_{\{1,2\}}}$ and for every $\alpha_{\{1,2,3\}'} \in \mathbb{N}_0^{\{1,2,3\}'}$ we can prove, in a similar way, that the map $f_{\alpha_{\{1,2,3\}'}}(\mathbf{z}_{\{1,2\}}, \cdot) : S_{\gamma_3} \rightarrow \mathbb{C}$, which belongs to $\mathcal{A}_{\mathbf{M}}(S_{\gamma_3})$, is null. So, we have $f_{\alpha_{\{1,2,3\}'}} \equiv 0$ in $S_{\gamma_{\{1,2,3\}'}}$ for every $\alpha_{\{1,2,3\}'} \in \mathbb{N}_0^{\{1,2,3\}'}$. After n steps, we conclude that $\text{TA}(f)$ is null. Now, condition (ii) together with Proposition 4.5 guarantees that $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is (s) quasi-analytic, so f will identically vanish, as desired. \square

We now give a consequence of Proposition 4.7.

Proposition 4.13. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence of positive real numbers, and $\gamma = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$. If $\int^{\infty} \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\underline{\gamma}}} dr$ does not converge, $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is quasi-analytic.

Proof:

Since $\underline{\gamma} \leq \gamma_j$ for every $j \in \mathcal{N}$, the integrals

$$\int^{\infty} \frac{\log T_{\mathbf{M}}(r)}{r^{1+1/\gamma_j}} dr$$

do not converge, so the classes $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$, $j \in \mathcal{N}$, are quasi-analytic, and we conclude the proof by the previous result. \square

If \mathbf{M} is a strongly regular sequence, we can get a consequence of Proposition 4.8.

Proposition 4.14. Let $n \in \mathbb{N}$, $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence, and $\gamma = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$ such that $0 < \underline{\gamma} < \gamma(\mathbf{M})$. Then, the class $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is non-quasi-analytic.

Proof:

There exists $j \in \mathcal{N}$ such that $\gamma_j = \underline{\gamma} < \gamma(\mathbf{M})$. According to Proposition 4.8, $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is non quasi-analytic. By Proposition 4.12, $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is also non quasi-analytic. \square

Under condition (15), we can prove the following equivalence, obtaining a new Watson's Lemma type result.

Proposition 4.15 (Second generalization of Watson's Lemma). Let \mathbf{M} be a strongly regular sequence that verifies condition (15), and let $n \in \mathbb{N}$ and $\gamma \in (0, \infty)^n$. The following statements are equivalent:

- (i) $\underline{\gamma} \geq \gamma(\mathbf{M})$.
- (ii) The class $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is quasi-analytic.

Proof:

(i) implies (ii) is the only thing left to prove. If (i) is fulfilled, for every $j \in \mathcal{N}$ we have $\gamma_j \geq \gamma(\mathbf{M})$. Proposition 4.9 guarantees that $\mathcal{A}_{\mathbf{M}}(S_{\gamma_j})$ is quasi-analytic, and we conclude by applying Proposition 4.12. \square

Remark 4.16. As it was indicated in Remark 4.11, we do not know whether the condition $\underline{\gamma} \geq \gamma(\mathbf{M})$ implies $\mathcal{A}_{\mathbf{M}}(S_{\gamma})$ is quasi-analytic without the additional assumption (15).

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