On strongly asymptotically developable functions and Borel-Ritt theorem

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Abstract. We show that the holomorphic functions on polysectors whose derivatives remain bounded on proper subpolysectors are precisely those strongly asymptotically developable in the sense of Majima. This fact allows us to solve two Borel-Ritt type interpolation problems from a functional-analysis-viewpoint.
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1 Introduction

It is well known that, for a function f, holomorphic on a sector S in the complex plane with vertex at 0, the existence of asymptotic expansion as the variable tends to 0 amounts to the boundedness of the derivatives of f on the bounded proper subsectors of S. Borel-Ritt theorem assures the existence of holomorphic functions on a given sector S admitting a prescribed asymptotic expansion at 0 in S. There are several classical proofs of this result in the literature (see, e.g., [Ol, Chapter 1, §9, p. 22], [Wa, Chapter III, §9.2, p. 43]). One of them (based on the ideas in [Ol, Chapter 4, §1.1, p. 106]; see Theorem 5.1 in this paper) has the particular feature that the solution is a function whose derivatives are in fact bounded on the unbounded proper subsectors of S. So, Borel-Ritt interpolation problem is solvable in a different setting.

The aim of this paper is to transfer this characterization and results to the case of strongly asymptotically developable holomorphic functions of several complex variables, as defined by Majima [Ma]. To this end, Section 3 is devoted to the study of the space $\mathcal{A}(S)$, consisting of the holomorphic functions on a polysector S of $\mathbb{C}^n$ whose derivatives remain bounded in the bounded proper subpolysectors of S; we give $\mathcal{A}(S)$ a natural Fréchet space topology, and prove that it is precisely the space of holomorphic functions on S strongly asymptotically developable at the origin. This equivalence allows to obtain many of the properties of these functions in an elementary way. The main ideas in this Section first appeared, for the Gevrey case, in the paper of Haraoka [Ha]; the results, in the present terms, come from the work of Hernández [He].

In Section 4, the corresponding Borel-Ritt problem in this context is stated: given a coherent family $\mathcal{F}$ (see Section 3 for the definition), does there exist $f \in \mathcal{A}(S)$ such that $TA(f) = \mathcal{F}$?

Majima [Ma 2, Part I, Theorem 3.1, p.35] gives a partial solution: for such an $\mathcal{F}$ and for a bounded proper subpolysector T of S, there exists $f \in \mathcal{A}(T)$ such that $TA(f) = \mathcal{F}$. Hernández [He] solves the problem, as initially stated, by a constructive method which strongly depends on the boundedness of the subpolysectors imposed in the definition of $\mathcal{A}(S)$. He considers the Fréchet space $\mathcal{A}(S, E)$ of holomorphic functions on a polysector S, with values in a Fréchet space E, and whose derivatives remain bounded on the proper bounded subpolysectors of S. After obtaining a solution to the problem in series form when S is a sector, he studies its properties in the particular case in which E is of the type $\mathcal{A}(U, E)$, U being a polysector; this, together with the fact that $\mathcal{A}(S, \mathcal{A}(U, E))$ and $\mathcal{A}(S \times U, E)$ are isomorphic, allows to apply an induction argument on the number of variables to conclude.

The solution in this paper is completely different, due to the following reasons. Section 5, of mainly theoretical interest, is devoted to obtain a Borel-Ritt type theorem in the framework of the space $\mathcal{B}(S)$ of holomorphic functions on an unbounded polysector S of $\mathbb{C}^n$ whose derivatives are bounded on the unbounded proper subpolysectors of S. This results corresponds, in the several variables case, to Theorem 5.1, and it was the motivation for the present work. It was not possible for us either to obtain a solution in series form in the one dimensional case, or to make a suitable study of the solution obtained in Theorem 5.1 (which remains valid when we consider the space $\mathcal{B}(S, E)$, E being a Fréchet space, instead of $\mathcal{B}(S)$) so that we might apply induction. So, a different approach was necessary. Functional-analysis techniques turned out to be fruitful not only in this situation, but also in a similar treatment for $\mathcal{A}(S)$.

2 Notation

For $n \in \mathbb{N}$, $n \geq 1$, put $N = \{1, 2, \ldots, n\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$ be two multiindices, m a natural number, and

$\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. We set

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n) \quad m\boldsymbol{\alpha} = (m\alpha_1, m\alpha_2, \ldots, m\alpha_n)$$

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad \boldsymbol{\alpha}! = \alpha_1!\, \alpha_2! \cdots \alpha_n!$$

$$\boldsymbol{\alpha} \leq \boldsymbol{\beta} \Leftrightarrow \alpha_j \leq \beta_j,\ j \in N \qquad \boldsymbol{\alpha} < \boldsymbol{\beta} \Leftrightarrow \alpha_j < \beta_j,\ j \in N$$

$$\mathbf{1} = (1, 1, \ldots, 1) \qquad \boldsymbol{e}_j = (\delta_{ij})_{i=1}^n$$

$$\boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n} \qquad |\boldsymbol{z}^{\boldsymbol{\alpha}}| = |\boldsymbol{z}|^{\boldsymbol{\alpha}} = |z_1|^{\alpha_1} |z_2|^{\alpha_2} \ldots |z_n|^{\alpha_n}$$

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}}.$$

If J is a nonempty subset of N, the number of elements of J will be $\#J$.

Let n be a natural number, $n \geq 1$, and consider, for $j = 1, 2, \ldots, n$, an open sector in $\mathbb{C}$ with vertex at the origin,

$$S_j = \{\, z \in \mathbb{C} : \theta_{1j} < \arg(z) < \theta_{2j} \,\}, \quad 0 < \theta_{2j} - \theta_{1j} \leq 2\pi.$$

We will call (unbounded open) polysector in $\mathbb{C}^n$ with vertex at $\mathbf{0}$ to any cartesian product of open sectors in $\mathbb{C}$ with vertex at 0, $S = \prod_{j=1}^n S_j \subset \mathbb{C}^n$.

We say a polysector T in $\mathbb{C}^n$ (with vertex at the origin) is a proper subpolysector of S if $T = \prod_{j=1}^n T_j$, with $\overline{T_j} \subset S_j \cup \{0\}$, $j = 1, 2, \ldots, n$. If

$$T_j = \{\, z \in \mathbb{C} : \varphi_{1j} < \arg(z) < \varphi_{2j},\ 0 < |z| < r_j \,\},$$

we say T is a bounded proper subpolysector of S.

If $J = \{j_1 < j_2 < \ldots < j_k\}$ is a nonempty subset of N and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we put $\boldsymbol{z}_J = (z_{j_1}, z_{j_2}, \ldots, z_{j_k})$. Let J and L be nonempty disjoint subsets of N. For $\boldsymbol{z}_J \in \mathbb{C}^J$ and $\boldsymbol{z}_L \in \mathbb{C}^L$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)$ represents the element of $\mathbb{C}^{J \cup L}$ verifying $(\boldsymbol{z}_J, \boldsymbol{z}_L)_J = \boldsymbol{z}_J$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)_L = \boldsymbol{z}_L$; we also write $J^c = N - J$, and for $j \in N$ we use j^c instead of $\{j\}^c$. In particular, we shall use these conventions for multiindices.

Finally, if $S = \prod_{j=1}^n S_j$ is a polysector of $\mathbb{C}^n$, $S_J = \prod_{j \in J} S_j \subset \mathbb{C}^J$.

3 Characterization of strongly asymptotically developable functions

Denote by $\mathcal{A}(S)$ the complex vector space consisting of the complex functions f defined and holomorphic in S, such that for each bounded proper subpolysector T of S and each $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$Q_{T,\boldsymbol{\alpha}}(f) = \sup\{\, |D^{\boldsymbol{\alpha}} f(\boldsymbol{z})| : \boldsymbol{z} \in T\,\} < +\infty.$$

Clearly, $\mathcal{A}(S)$ is closed under product and differentiation. We shall adopt as topology in $\mathcal{A}(S)$ the one generated by the family of seminorms $\{Q_{T,\boldsymbol{\alpha}}\}$; this makes $\mathcal{A}(S)$ a Fréchet space.

Let $f \in \mathcal{A}(S)$. Since all its derivatives are bounded on bounded proper subpolysectors of S, Barrow's formula implies that they are also lipschitzian. Hence, if $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, we can define a function from S_{J^c} to $\mathbb{C}$ by

$$(1) \qquad f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_J \to \boldsymbol{0} \\ \boldsymbol{z}_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \boldsymbol{0}_{J^c})} f(\boldsymbol{z})}{\boldsymbol{\alpha}_J!}, \quad \boldsymbol{z}_{J^c} \in S_{J^c},$$

for any subpolysector T_J of S_J; the limit is uniform on bounded proper subpolysectors of S_{J^c} (whenever $J \neq N$), what implies that $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$ (setting $\mathcal{A}(S_{N^c}) = \mathbb{C}$). Also, the map from $\mathcal{A}(S)$ to $\mathcal{A}(S_{J^c})$ sending f to $f_{\boldsymbol{\alpha}_J}$ is continuous.

Accordingly, we may associate to f a family

$$\mathcal{F}(f) = \{\, f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \,\},$$

which we call the *derived family* for f. The aforementioned uniformity in the limits defining its elements entails

Proposition 3.1 (Coherence conditions) *Let $f \in \mathcal{A}(S)$, and let $\mathcal{F}(f)$ be its derived family. The following properties hold:*

i) For every disjoint nonempty subsets J and L of N, with $J \cup L \neq N$, for every $\boldsymbol{\alpha}_J \in \mathbb{N}^J$ and $\boldsymbol{\alpha}_L \in \mathbb{N}^L$, and for every proper subpolysector T_L of S_L,

$$\lim_{\substack{\boldsymbol{z}_L \to \boldsymbol{0} \\ \boldsymbol{z}_L \in T_L}} \frac{D^{(\boldsymbol{\alpha}_L, \boldsymbol{0}_{(J\cup L)^c})} f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_L!} = f_{(\boldsymbol{\alpha}_J, \boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)^c}),$$

uniformly on the bounded proper subpolysectors of $S_{(J\cup L)^c}$.

ii) For each nonempty subset J *of* N, *for each multiindex* $\boldsymbol{\alpha} \in \mathbb{N}^n$ *and for each proper subpolysector* T_{J^c} *of* S_{J^c},

$$\lim_{\substack{\boldsymbol{z}_{J^c}\to\boldsymbol{0}\\ \boldsymbol{z}_{J^c}\in T_{J^c}}} \frac{D^{\boldsymbol{\alpha}_{J^c}} f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_{J^c}!} = f_{\boldsymbol{\alpha}} \in \mathbb{C}.$$

Hereafter, we will say that a family

$$\mathcal{F} = \{ f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \},$$

or briefly $\mathcal{F} = \{ f_{\boldsymbol{\alpha}_J} \}$, is coherent if it verifies i) and ii).

The concept of strong asymptotic development was established by Majima (see [Ma]) in order to study solutions for integrable connections with irregular singular points. Let f be a complex function defined and holomorphic in a polysector S of $\mathbb{C}^n$ with vertex at $\mathbf{0}$. We say that f is strongly asymptotically developable at $\mathbf{0}$ if there exists a family

$$\mathcal{F} = \left\{ f_{\boldsymbol{\alpha}_J} : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \right\},$$

where $f_{\boldsymbol{\alpha}_J}$ is a holomorphic function from S_{J^c} to $\mathbb{C}$, when $J \neq N$, and $f_{\boldsymbol{\alpha}_J} \in \mathbb{C}$ when $J = N$, satisfying the following bounds: if we define

$$App_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z}) = \sum_{\emptyset\neq J\subset N} (-1)^{\#J+1} \sum_{\substack{\boldsymbol{\beta}_J\in\mathbb{N}^J\\ \boldsymbol{\beta}_J\leq\boldsymbol{\alpha}_J-\boldsymbol{1}_J}} f_{\boldsymbol{\beta}_J}(\boldsymbol{z}_{J^c})\boldsymbol{z}_J^{\boldsymbol{\beta}_J}, \quad \boldsymbol{\alpha}\in\mathbb{N}^n,\ \boldsymbol{z}\in S,$$

then for every bounded proper subpolysector T of S and for every $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$\sup\left\{ \left|\frac{f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\right| : \boldsymbol{z}\in T \right\} < \infty.$$

Under these conditions, $\mathcal{F}$ will be called total family of strongly asymptotic expansion associated to f, and will be denoted by $TA(f)$. For $\boldsymbol{\alpha} \in \mathbb{N}^n$, the function $App_{\boldsymbol{\alpha}}(\mathcal{F})$, defined and holomorphic from S to $\mathbb{C}$, is called the approximate function of order $\boldsymbol{\alpha}$ corresponding to the family $\mathcal{F}$.

Theorem 3.2 *Let* f *be a holomorphic function from* S *to* $\mathbb{C}$. *Then,* f *is strongly asymptotically developable at* $\mathbf{0}$ *in* S *if and only if* $f \in \mathcal{A}(S)$. *If this is the case,* $\mathcal{F}(f) = TA(f)$.

Proof: Assume f is strongly asymptotically developable. Consider a proper bounded subpolysector T of S and $\boldsymbol{\alpha} \in \mathbb{N}^n$. We may take a new proper bounded subpolysector T_1 of S such that T is proper in T_1, which allows us to determine $r > 0$ such that for every $\boldsymbol{z} \in T$, the closed polydisc centered at $\boldsymbol{z}$ with polyradius $\boldsymbol{r}(\boldsymbol{z}) = (r|z_1|, r|z_2|, \ldots, r|z_n|) \in (0,\infty)^n$, denoted by $\overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})$, is contained in T_1. If $\boldsymbol{\omega}$ belongs to the distinguished boundary $\partial_0\overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})$, then $|\boldsymbol{\omega}|^{\boldsymbol{\alpha}} \leq (1+r)^{|\boldsymbol{\alpha}|}|\boldsymbol{z}|^{\boldsymbol{\alpha}}$ and $|\boldsymbol{\omega}-\boldsymbol{z}|^{\boldsymbol{\alpha}+\boldsymbol{1}} = r^{|\boldsymbol{\alpha}|+n}|\boldsymbol{z}|^{\boldsymbol{\alpha}+\boldsymbol{1}}$. As f admits strongly asymptotic expansion at $\boldsymbol{0}$, there exists $C_{T_1,\boldsymbol{\alpha}} > 0$ such that

$$|f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(TA(f))(\boldsymbol{z})| \leq C_{T_1,\boldsymbol{\alpha}}|\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T_1.$$

Since $D^{\boldsymbol{\alpha}}App_{\boldsymbol{\alpha}}(TA(f)) \equiv 0$ on S, we can apply Cauchy's integral formula to obtain that, for every $\boldsymbol{z} \in T$,

$$\begin{aligned}|D^{\boldsymbol{\alpha}}f(\boldsymbol{z})| &= \left|\frac{\boldsymbol{\alpha}!}{(2\pi i)^n}\int_{\partial_0\overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})}\frac{f(\boldsymbol{\omega}) - App_{\boldsymbol{\alpha}}(TA(f))(\boldsymbol{\omega})}{(\boldsymbol{\omega}-\boldsymbol{z})^{\boldsymbol{\alpha}+\boldsymbol{1}}}\,d\boldsymbol{\omega}\right| \\ &\leq \boldsymbol{\alpha}!\,C_{T_1,\boldsymbol{\alpha}}\Big(\frac{1+r}{r}\Big)^{|\boldsymbol{\alpha}|} < \infty,\end{aligned}$$

and we deduce that $f \in \mathcal{A}(S)$.

Conversely, let $f \in \mathcal{A}(S)$ and $\mathcal{F}(f)$ be its derived family. The error formula

$$f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(\mathcal{F}(f))(\boldsymbol{z}) = \prod_{\substack{j=1\\ \alpha_j\neq 0}}^{n}\Big(\int_0^{z_j}dt_{j,1}\int_0^{t_{j,1}}dt_{j,2}\cdots\int_0^{t_{j,\alpha_j-1}}dt_{j,\alpha_j}\Big)\,D^{\boldsymbol{\alpha}}f(t_{1,\alpha_1},t_{2,\alpha_2},\ldots,t_{n,\alpha_n}),$$

was given by Haraoka (cf. [Ha]; a proof of it, valid in our setting, can be found in a paper by Zurro [Zu]). Consider a proper bounded subpolysector T of S. Since f belongs to $\mathcal{A}(S)$, for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\sup_{\boldsymbol{z}\in T}|D^{\boldsymbol{\alpha}}f(\boldsymbol{z})| = C_{T,\boldsymbol{\alpha}} < +\infty$. Then, if $\boldsymbol{z} \in T$,

$$|f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(\mathcal{F}(f))(\boldsymbol{z})| \leq \frac{|\boldsymbol{z}|^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}C_{T,\boldsymbol{\alpha}},$$

so that f admits strongly asymptotic expansion at $\boldsymbol{0}$, and $TA(f) = \mathcal{F}(f)$.□

Some remarks are in order. The uniqueness of $TA(f)$ follows from the expressions given in (1). So, the approximate functions will be denoted henceforth as $App_{\boldsymbol{\alpha}}(f)$, $\boldsymbol{\alpha} \in \mathbb{N}^n$. For $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$; thus, the elements of the total family are strongly asymptotically developable, and from the coherence conditions it becomes obvious that

$$TA(f_{\boldsymbol{\alpha}_J}) = \{\, f_{(\boldsymbol{\alpha}_J,\boldsymbol{\beta}_L)} : \emptyset \neq L \subset J^c,\ \boldsymbol{\beta}_L \in \mathbb{N}^L \,\}.$$

We also note that the notion of consistent family given by Majima (see [Ma2, Part I, p.25]) is now seen to be equivalent to that of coherent family. Finally, it is evident that the family of seminorms $\{P_{T,\boldsymbol{\alpha}}\}$, defined on $\mathcal{A}(S)$ for every bounded proper subpolysector T of S and $\boldsymbol{\alpha} \in \mathbb{N}^n$ as

$$P_{T,\boldsymbol{\alpha}}(f) = \sup\Big\{ \Big|\frac{f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| : \boldsymbol{z} \in T \Big\},$$

generates in $\mathcal{A}(S)$ the natural topology. æ

4 Interpolation problem of Borel-Ritt type in $\mathcal{A}(S)$.

If $f \in \mathcal{A}(S)$, $TA(f)$ is coherent. Thus, the following Borel-Ritt type problem arises:

Given a coherent family $\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \}$, does there exist $f \in \mathcal{A}(S)$ such that $TA(f) = \mathcal{F}$?

In order to solve this problem we now give another two equivalent settings, obtained by changing the initially given data to interpolate. This will also let us go deeper into the relations linking the different elements in the concept of strongly asymptotic expansion.

First approach: If $f \in \mathcal{A}(S)$, we call first order family associated to f to

$$TA'(f) = \{\, f_{m_{\{j\}}} \in \mathcal{A}(S_{j^c}) : j \in N,\ m \in \mathbb{N} \,\},$$

i.e., the subfamily of $TA(f)$ consisting of those elements in $n-1$ variables. For convenience, we write f_{jm} instead of $f_{m_{\{j\}}}$. $TA'(f)$ satisfies "first order" coherence conditions:

a) Let $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $j, \ell \in N$. For each bounded proper subpolysector T of S,

$$\lim_{\substack{\boldsymbol{z}_{j^c} \to 0 \\ \boldsymbol{z}_{j^c} \in T_{j^c}}} \frac{D^{\boldsymbol{\alpha}_{j^c}} f_{j\alpha_j}(\boldsymbol{z}_{j^c})}{\boldsymbol{\alpha}_{j^c}!} = \lim_{\substack{\boldsymbol{z}_{\ell^c} \to 0 \\ \boldsymbol{z}_{\ell^c} \in T_{\ell^c}}} \frac{D^{\boldsymbol{\alpha}_{\ell^c}} f_{\ell\alpha_\ell}(\boldsymbol{z}_{\ell^c})}{\boldsymbol{\alpha}_{\ell^c}!}.$$

And, if $n \geq 3$,

b) Let L be a proper subset of N consisting of at least two elements, $\boldsymbol{\alpha}_L \in \mathbb{N}^L$ and T_L a bounded proper subpolysector of S_L. For every $j, \ell \in L$,

$$\lim_{\substack{\boldsymbol{z}_{L-\{j\}} \to 0 \\ \boldsymbol{z}_{L-\{j\}} \in T_{L-\{j\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{j\}}, \boldsymbol{0}_{L^c})} f_{j\alpha_j}(\boldsymbol{z}_{j^c})}{\boldsymbol{\alpha}_{L-\{j\}}!} = \lim_{\substack{\boldsymbol{z}_{L-\{\ell\}} \to 0 \\ \boldsymbol{z}_{L-\{\ell\}} \in T_{L-\{\ell\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{\ell\}}, \boldsymbol{0}_{L^c})} f_{\ell\alpha_\ell}(\boldsymbol{z}_{\ell^c})}{\boldsymbol{\alpha}_{L-\{\ell\}}!},$$

uniformly on the bounded proper subpolysectors of S_{L^c}.

It turns out that $TA'(f)$, under these first order conditions, determines $TA(f)$ uniquely. Indeed, the case $n = 2$ is obvious; otherwise, let J be a subset of N consisting of at least two elements, and let $\boldsymbol{\alpha}_J \in \mathbb{N}^J$. Choose $j \in J$; $f_{\boldsymbol{\alpha}_J}$ can be recovered as

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_{J-\{j\}} \to \boldsymbol{0} \\ \boldsymbol{z}_{J-\{j\}} \in T_{J-\{j\}}}} \frac{D^{(\boldsymbol{\alpha}_{J-\{j\}}, \boldsymbol{0}_{J^c})} f_{j\alpha_j}(\boldsymbol{z}_{j^c})}{\boldsymbol{\alpha}_{J-\{j\}}!}, \quad \boldsymbol{z}_{J^c} \in S_{J^c}, \tag{2}$$

$T_{J-\{j\}}$ being a bounded proper subpolysector of $S_{J-\{j\}}$. In fact, if we consider a family

$$\mathcal{F}' = \{\, f_{jm} \in \mathcal{A}(S_{j^c}) : j \in N,\ m \in \mathbb{N} \,\}$$

under the previous first order coherence conditions (henceforth, we will say that $\mathcal{F}' = \{\, f_{jm} \,\}$ is a coherent first order family), the relations in (2) define, with no ambiguity, a function $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$, and we may construct a family $\mathcal{F} = \{\, f_{\boldsymbol{\alpha}_J} \,\}$ that, applying Proposition 3.1 to the functions f_{jm}, is seen to be coherent. So, Borel-Ritt problem may be rewritten as follows:

Given a coherent first order family $\mathcal{F}'$, prove the existence of a function $f \in \mathcal{A}(S)$ such that $TA'(f) = \mathcal{F}'$.

Second approach: Consider the family of approximate functions for $f \in \mathcal{A}(S)$, $App(f) = (App_{\boldsymbol{\alpha}}(f))_{\boldsymbol{\alpha} \in \mathbb{N}^n}$. Of course, the knowledge of $TA(f)$ entails that of $App(f)$; the converse is also true, since, for $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, we have, from the coherence conditions, that

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_J \to \boldsymbol{0} \\ \boldsymbol{z}_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \boldsymbol{0}_{J^c})} App_{(\boldsymbol{\alpha}_J + \boldsymbol{1}_J, \boldsymbol{0}_{J^c})}(f)(\boldsymbol{z}_J, \boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_J!},$$

T_J being a bounded proper subpolysector of S_J. Next, observe that, for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, $App_{\boldsymbol{\alpha}}(f) \in \mathcal{A}_{\boldsymbol{\alpha}}$, where

$$\mathcal{A}_{\boldsymbol{\alpha}} = \{\, g \in \mathcal{A}(S) \text{ such that there exists } h \in \mathcal{A}(S) \text{ with } g = App_{\boldsymbol{\alpha}}(h) \,\}.$$

Moreover, if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ and $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$, a straightforward calculation gives

$$App_{\boldsymbol{\beta}}(App_{\boldsymbol{\alpha}}(f)) = App_{\boldsymbol{\beta}}(f),$$

and so, if we call $\mathcal{P} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$, then $App(f) \in \mathcal{D}$, where

$$\mathcal{D} = \{ (g_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \mathcal{P} : \text{if } \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n \text{ and } \boldsymbol{\beta} \leq \boldsymbol{\gamma}, \text{ then } App_{\boldsymbol{\beta}}(g_{\boldsymbol{\gamma}}) = g_{\boldsymbol{\beta}} \}.$$

Conversely, if we begin with $(g_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \mathcal{D}$, we may construct a family $\mathcal{F} = \{ f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \}$ by defining

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_J \to \boldsymbol{0} \\ \boldsymbol{z}_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \boldsymbol{0}_{J^c})} g_{(\boldsymbol{\alpha}_J + \boldsymbol{1}_J, \boldsymbol{0}_{J^c})}(\boldsymbol{z}_J, \boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_J!},$$

and it may be easily proved that $\mathcal{F}$ is coherent. Hence, our problem may also be expressed in these terms:

Given a family $\mathcal{I} \in \mathcal{D}$, does there exist a function $f \in \mathcal{A}(S)$ such that $App(f) = \mathcal{I}$?

We shall answer the problem in the affirmative in this second approach, while the first one will come into play in the last section.

Let us define the map ψ from $\mathcal{A}(S)$ to $\mathcal{D}$, which takes $f \in \mathcal{A}(S)$ to $\psi(f) = App(f)$. Solving the problem amounts to proving the surjectivity of ψ.

For each $\boldsymbol{\alpha} \in \mathbb{N}^n$, let us provide the space $\mathcal{A}_{\boldsymbol{\alpha}}$ with the topology of subspace of $\mathcal{A}(S)$. $\mathcal{A}_{\boldsymbol{\alpha}}$ is a Fréchet space. $\mathcal{P} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$ is given the product topology, and $\mathcal{D} \subset \mathcal{P}$ the subspace topology. Again, $\mathcal{D}$ is a Fréchet space.

The map ψ, from $\mathcal{A}(S)$ to $\mathcal{D}$, is continuous. Indeed, it suffices to prove that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, the map $\psi_{\boldsymbol{\alpha}}$ from $\mathcal{A}(S)$ to $\mathcal{A}_{\boldsymbol{\alpha}}$ defined by

$$\psi_{\boldsymbol{\alpha}}(f) = App_{\boldsymbol{\alpha}}(f), \qquad f \in \mathcal{A}(S),$$

is continuous. Suppose a sequence $\{f_\ell\}_{\ell=1}^\infty$ of elements of $\mathcal{A}(S)$ converges to $g \in \mathcal{A}(S)$ and $\{App_{\boldsymbol{\alpha}}(f_\ell)\}_{\ell=1}^\infty$ converges to $h \in \mathcal{A}_{\boldsymbol{\alpha}}$; then, the continuity of the map from $\mathcal{A}(S)$ to $\mathcal{A}(S_{J^c})$ sending f to $f_{\boldsymbol{\alpha}_J}$ implies that

$$App_{\boldsymbol{\alpha}}(g)(\boldsymbol{z}) = \lim_{\ell\to\infty} App_{\boldsymbol{\alpha}}(f_\ell)(\boldsymbol{z}), \quad \boldsymbol{z} \in S.$$

Since the convergence in $\mathcal{A}_{\boldsymbol{\alpha}}$ assures the pointwise one, this last limit equals $h(\boldsymbol{z})$, and we conclude with the Closed Graph Theorem.

We may then apply the following result (see [Ho, Chapter 3, §13, Proposition 3 and its Corollary, p. 263-264]):

ψ is surjective if and only if its transpose ${}^t\psi$, from $\mathcal{D}'$ to $\mathcal{A}(S)'$, is injective and ${}^t\psi(\mathcal{D}')$ is $\sigma(\mathcal{A}(S)', \mathcal{A}(S))$-closed in $\mathcal{A}(S)'$. Also, ${}^t\psi$ is injective if and only if $\psi(\mathcal{A}(S))$ is dense in $\mathcal{D}$.

Let us show that the ψ-image of $\mathcal{A}(S)$ is a dense subset of $\mathcal{D}$: if $\mathcal{I} = (f_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \mathcal{D}$, a neighbourhood V of $\mathcal{I}$ is the product of neighbourhoods $V_{\boldsymbol{\alpha}}$ of $f_{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha} \in \mathbb{N}^n$, being $V_{\boldsymbol{\alpha}} = \mathcal{A}_{\boldsymbol{\alpha}}$ except for finitely many multiindices, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$. Consider the multiindex

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \cdots + \boldsymbol{\alpha}_m \in \mathbb{N}^n,$$

and the function $f_{\boldsymbol{\alpha}_0} \in \mathcal{A}_{\boldsymbol{\alpha}_0} \subset \mathcal{A}(S)$. Since $\boldsymbol{\alpha}_\ell \leq \boldsymbol{\alpha}_0$ for $\ell = 1, 2, \ldots, m$, $App_{\boldsymbol{\alpha}_\ell}(f_{\boldsymbol{\alpha}_0}) = f_{\boldsymbol{\alpha}_\ell}$, and so, $\psi(f_{\boldsymbol{\alpha}_0}) = (App_{\boldsymbol{\alpha}}(f_{\boldsymbol{\alpha}_0}))_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in V$. Hence, ${}^t\psi$ is injective.

In order to conclude, we shall prove that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{A}(S)'$. The three following results are needed.

Proposition 4.1 *Let $L \in \mathcal{A}(S)'$ belong to the weak closure of ${}^t\psi(\mathcal{D}')$. Then,*

$$\bigcap_{\boldsymbol{\alpha}\in\mathbb{N}^n} Ker(\psi_{\boldsymbol{\alpha}}) \subset Ker(L),$$

i.e., if $f \in \mathcal{A}(S)$ is such that $App_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$, then $L(f) = 0$.

Proof: Let $f \in \bigcap_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathrm{Ker}(\psi_{\boldsymbol{\alpha}})$. Consider, for $p \in \mathbb{N}$, $p \geq 1$, the weak neighbourhood V_p of L,

$$V_p = \{ M \in \mathcal{A}(S)' : |M(f) - L(f)| < \frac{1}{p} \}.$$

Since $V_p \cap {}^t\psi(\mathcal{D}') \neq \emptyset$, there exists $\xi_p \in \mathcal{D}'$ such that $|{}^t\psi(\xi_p)(f) - L(f)| < \frac{1}{p}$. $\mathcal{D}$ is provided with the subspace topology of $\mathcal{P} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$, so we have $\mathcal{P}' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}} \subset \mathcal{D}'$, and it makes sense to define the map $\overline{\psi}$ from $\bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}}$ to $\mathcal{A}(S)'$ given by

$$\overline{\psi}(\xi) = {}^t\psi(\xi|_{\mathcal{D}}) = {}^t\psi \circ \mu(\xi), \quad \xi \in \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}},$$

where μ represents the restriction operator to $\mathcal{D}$ of continuous linear functionals defined on $\mathcal{P}$. By virtue of the theorem of Hahn-Banach, μ is surjective. Thus, there exists $\eta_p \in \mathcal{P}' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}}$ such that $\mu(\eta_p) = \xi_p$. Then,

$$\overline{\psi}(\eta_p) = {}^t\psi \circ \mu(\eta_p) = {}^t\psi(\xi_p).$$

We may write, for an adequate $\boldsymbol{\alpha}_0 \in \mathbb{N}^n$, $\eta_p = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n,\boldsymbol{\alpha}\leq\boldsymbol{\alpha}_0} \eta_{\boldsymbol{\alpha}}$, $\eta_{\boldsymbol{\alpha}} \in \mathcal{A}'_{\boldsymbol{\alpha}}$. For $g \in \mathcal{A}(S)$,

$$\overline{\psi}(\eta_{\boldsymbol{\alpha}})(g) = {}^t\psi \circ \mu(\eta_{\boldsymbol{\alpha}})(g) = {}^t\psi(\eta_{\boldsymbol{\alpha}}|_{\mathcal{D}})(g) = \eta_{\boldsymbol{\alpha}}|_{\mathcal{D}} \circ \psi(g)$$

$$= \eta_{\boldsymbol{\alpha}}|_{\mathcal{D}}((App_{\boldsymbol{\alpha}}(g))_{\boldsymbol{\alpha}\in\mathbb{N}^n}) = \eta_{\boldsymbol{\alpha}}(App_{\boldsymbol{\alpha}}(g)) = \eta_{\boldsymbol{\alpha}} \circ \psi_{\boldsymbol{\alpha}}(g).$$

Since $\psi_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha}$, we get that for each $p \in \mathbb{N}$, $p \geq 1$,

$$|L(f)| = |L(f) - \sum_{\boldsymbol{\alpha}\leq\boldsymbol{\alpha}_0} \eta_{\boldsymbol{\alpha}} \circ \psi_{\boldsymbol{\alpha}}(f)| = |L(f) - \overline{\psi}(\eta_p)(f)|$$

$$= |L(f) - {}^t\psi(\xi_p)(f)| < \frac{1}{p},$$

and hence, $L(f) = 0$. □

Proposition 4.2 *Let $L \in \mathcal{A}(S)'$ satisfy the following property:*
If $f \in \mathcal{A}(S)$ and for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ it holds $App_{\boldsymbol{\alpha}}(f) = 0$, then $L(f) = 0$.
Then there exists $r \in \mathbb{N}$, $r \geq 1$, such that if $f \in \mathcal{A}(S)$ and for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| \leq r+1$ we have $App_{\boldsymbol{\alpha}}(f) = 0$, then $L(f) = 0$.

Proof: Consider, for $j \in N$, a sequence of bounded proper subsectors of S_j, $\{T_{jr}\}_{r\in\mathbb{N}}$, such that:
i) T_{jr} is a bounded proper subpolysector of $T_{j,r+1}$, $r \in \mathbb{N}$.
ii) If K is a compact subset of S_j, there exists $r \in \mathbb{N}$ such that $K \subset T_{jr}$.
iii) T_{j1} has nonempty interior.
Define, for $r \in \mathbb{N}$, $T_r = \prod_{j\in N} T_{jr}$, and the seminorm p_r on $\mathcal{A}(S)$ given by

$$p_r(f) = \sup_{\boldsymbol{\alpha}\in\mathbb{N}^n,0\leq|\boldsymbol{\alpha}|\leq r} \sup_{\boldsymbol{z}\in T_r} \Big|\frac{f(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big|.$$

The increasing sequence of seminorms $\{p_r\}_{r\in\mathbb{N}}$ defines the topology of $\mathcal{A}(S)$. Hence, the sets $V_{\delta,r} = \{\, f \in \mathcal{A}(S) : p_r(f) < \delta \,\}$, $r \in \mathbb{N}$, $\delta > 0$, form a fundamental system of neighbourhoods of the origin in $\mathcal{A}(S)$, and given

$\varepsilon = 1$, there exist $\delta > 0$ and $r \in \mathbb{N}$ such that, if $f \in V_{\delta,r}$, then $|L(f)| < 1$. Notice that $V_{\delta,r'} \subset V_{\delta,r}$ if $r' > r$, so that we can assume, without loss of generality, that $r \geq 1$.

Let $g \in \mathcal{A}(S)$ be such that for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| \leq r+1$ we have $App_{\boldsymbol{\alpha}}(g) = 0$. We will prove that $L(g) = 0$ by constructing a sequence $\{g_k\}_{k=1}^{\infty}$ of elements of $\mathcal{A}(S)$ such that $L(g_k) = L(g)$ for all k, and $\lim_{k\to\infty} L(g_k) = 0$.

For $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq r+1$, there exists $C_{\boldsymbol{\alpha},r} > 0$ such that

$$\sup_{\boldsymbol{z}\in T_r} \left|\frac{g(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\right| \leq C_{\boldsymbol{\alpha},r}. \tag{3}$$

The following fact can be easily proved (cf. [Co]): given a sector U in $\mathbb{C}$ with vertex at the origin, there exists a holomorphic function β from U to $\mathbb{C}$ satisfying:

i) $\sup_{z\in U} |\beta(z)| < \infty$;

ii) For each $j \in \mathbb{N}$, $\displaystyle\lim_{\substack{z\to 0\\ z\in U}} \frac{|\beta(z)-1|}{|z|^j} = 0$, and $\displaystyle\sup_{z\in U} \frac{|\beta(z)-1|}{|z|^j} < \infty$;

iii) There exist $H, R > 0$ such that for $z \in U$ with $|z| \geq R$, we have

$$|\beta(z)| \leq \frac{H}{|z|^{r+1}}.$$

Without loss of generality, we may assume that the sectors S_j, $j = 1, \ldots, n$, have the positive real semiaxis as their symmetry semiaxis. Then it is possible to consider a function β as above, holomorphic in a sector U de $\mathbb{C}$ such that $S_j \subset U$, $j \in N$. Define the function γ from S to $\mathbb{C}$ given, for $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in S$, by

$$\gamma(\boldsymbol{z}) = 1 - (-1)^n \prod_{j\in N} (\beta(z_j) - 1) = \sum_{\emptyset\neq J\subset N} (-1)^{\#J+1} \prod_{j\in J} \beta(z_j).$$

γ is obviously holomorphic in S, and it satisfies:

i') $\sup_{\boldsymbol{z}\in S} |\gamma(\boldsymbol{z})| \leq 1 + \prod_{j\in N}(1 + \sup_{z_j\in S_j} |\beta(z_j)|) < \infty$;

ii') For every $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\displaystyle\lim_{\substack{\boldsymbol{z}\to 0\\ \boldsymbol{z}\in S}} \left|\frac{\gamma(\boldsymbol{z})-1}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\right| = 0$, and $\displaystyle\sup_{\boldsymbol{z}\in T_\ell} \left|\frac{\gamma(\boldsymbol{z})-1}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\right| < \infty$, $\ell \in \mathbb{N}$; hence $\gamma \in \mathcal{A}(S)$.

For $k \in \mathbb{N}$, $k \geq 1$, we define the functions

$$\gamma_k(\boldsymbol{z}) = \gamma(k\boldsymbol{z}) \quad \text{and} \quad g_k(\boldsymbol{z}) = \gamma_k(\boldsymbol{z})g(\boldsymbol{z}), \quad \boldsymbol{z} \in S.$$

If $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $\ell \in \mathbb{N}$, we have

$$\sup_{\boldsymbol{z}\in T_\ell} \Big|\frac{g_k(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(g)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| = \sup_{\boldsymbol{z}\in T_\ell} \Big|\frac{\gamma_k(\boldsymbol{z})g(\boldsymbol{z}) - g(\boldsymbol{z}) + g(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(g)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big|$$

$$\leq \sup_{\boldsymbol{z}\in T_\ell} |g(\boldsymbol{z})| \sup_{\boldsymbol{z}\in T_\ell} \Big|\frac{\gamma_k(\boldsymbol{z}) - 1}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| + \sup_{\boldsymbol{z}\in T_\ell} \Big|\frac{g(\boldsymbol{z}) - App_{\boldsymbol{\alpha}}(g)(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| < \infty,$$

where we have used ii'). So, $g_k \in \mathcal{A}(S)$, $k \geq 1$, and $App_{\boldsymbol{\alpha}}(g_k) = App_{\boldsymbol{\alpha}}(g)$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, which leads to $L(g_k) = L(g)$, $k \geq 1$.

Next we will prove that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| \leq r$ there exists a constant $C_0(\boldsymbol{\alpha}) > 0$ such that

$$\sup_{\boldsymbol{z}\in T_r} \Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| < \frac{C_0(\boldsymbol{\alpha})}{\sqrt{k}},$$

whenever $k > R^2$ (R being the constant introduced in iii)).

Fix $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| \leq r$, and let $\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \boldsymbol{e}_j$, $j \in N$. As $|\overline{\boldsymbol{\alpha}}| = |\boldsymbol{\alpha}| + 1 \leq r + 1$, by (3) and i') we have that for every $\boldsymbol{z} \in T_r$

$$\Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| = \Big|\frac{\gamma_k(\boldsymbol{z})g(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| \leq C_\gamma \Big|\frac{g(\boldsymbol{z})}{\boldsymbol{z}^{\overline{\boldsymbol{\alpha}}}}\Big| |\boldsymbol{z}^{\boldsymbol{e}_j}| \leq C_\gamma C_{\overline{\boldsymbol{\alpha}},r} |z_j|.$$

If for any $j \in N$ we have $|z_j| \leq \frac{1}{\sqrt{k}}$, then

$$\Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| \leq C_\gamma C_{\boldsymbol{\alpha}+\boldsymbol{e}_j,r} \frac{1}{\sqrt{k}} \leq C_\gamma (\max_{j\in N} C_{\boldsymbol{\alpha}+\boldsymbol{e}_j,r}) \frac{1}{\sqrt{k}} \leq \frac{C_1(\boldsymbol{\alpha})}{\sqrt{k}}.$$

If $|z_j| > \frac{1}{\sqrt{k}}$ for all $j \in N$, suppose that $k > R^2$. We have $|kz_j| > \sqrt{k} > R$, and so

$$|\beta(kz_j)| \leq \frac{H}{|kz_j|^{r+1}}.$$

On the other hand, $|\boldsymbol{z}^{\boldsymbol{\alpha}}| = \prod_{j\in N} |z_j|^{\alpha_j} > (k^{-1/2})^{\alpha_1+\cdots+\alpha_n} = k^{-|\boldsymbol{\alpha}|/2}$. Applying again (3) and the definition of γ, we may write

$$\Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| = \frac{1}{|\boldsymbol{z}^{\boldsymbol{\alpha}}|} |\gamma_k(\boldsymbol{z})g(\boldsymbol{z})| \leq C_{0,r} \frac{1}{|\boldsymbol{z}^{\boldsymbol{\alpha}}|} |\gamma(k\boldsymbol{z})|$$

$$\leq \frac{C_{0,r}}{|\boldsymbol{z}^{\boldsymbol{\alpha}}|} \sum_{\substack{J\subset N\\ J\neq\emptyset}} \Big(\prod_{j\in J} |\beta(kz_j)|\Big) \leq \frac{C_{0,r}}{|\boldsymbol{z}^{\boldsymbol{\alpha}}|} \sum_{\substack{J\subset N\\ J\neq\emptyset}} \Big(\prod_{j\in J} \frac{H}{|kz_j|^{r+1}}\Big)$$

$$\leq C_{0,r} \sum_{J\subset N, J\neq\emptyset} \frac{H^{\#J}}{k^{-|\boldsymbol{\alpha}|/2} k^{(r+1)(\#J)/2}} \leq \frac{(2^n-1)H_1 C_{0,r}}{\sqrt{k}} = \frac{C_2(\boldsymbol{\alpha})}{\sqrt{k}},$$

because $(r+1)(\#J) - |\boldsymbol{\alpha}| \geq r+1-|\boldsymbol{\alpha}| \geq 1$, and with $H_1 = \max_{\emptyset\neq J\subset N} H^{\#J}$. Therefore, there exists $C_0(\boldsymbol{\alpha}) = \max(C_1(\boldsymbol{\alpha}), C_2(\boldsymbol{\alpha})) > 0$ such that, if $k > R^2$, then

$$\sup_{\boldsymbol{z}\in T_r} \Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| < \frac{C_0(\boldsymbol{\alpha})}{\sqrt{k}},$$

and thus,

$$p_r(g_k) = \sup_{0\leq|\boldsymbol{\alpha}|\leq r} \sup_{\boldsymbol{z}\in T_r} \Big|\frac{g_k(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\Big| < \frac{\max_{0\leq|\boldsymbol{\alpha}|\leq r} C_0(\boldsymbol{\alpha})}{\sqrt{k}} = \frac{C_0}{\sqrt{k}},$$

or, in another way, $p_r(k^{1/4}g_k) < \frac{C_0}{k^{1/4}}$, $k > R^2$. There exists $k_1 \in \mathbb{N}$, $k_1 > R^2$, such that, if $k \geq k_1$, $p_r(k^{1/4}g_k) < \delta$, therefore $|L(k^{1/4}g_k)| < 1$, and $L(g) = \lim_{k\to\infty} L(g_k) = 0$. □

The existence of the natural number r in this Proposition ensures the existence of a multiindex $\boldsymbol{\beta} \in \mathbb{N}^n$ (e.g., $\boldsymbol{\beta} = (r+1, r+1, \ldots, r+1)$) such that if $f \in \mathcal{A}(S)$ and for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ we have $App_{\boldsymbol{\alpha}}(f) = 0$, then $L(f) = 0$.

Proposition 4.3 *Let $L \in \mathcal{A}(S)'$ such that there exists $\boldsymbol{\beta} \in \mathbb{N}^n$ with the following property:*

For every function $f \in \mathcal{A}(S)$ such that $App_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$, it holds $L(f) = 0$.
Then, there exists a functional $H \in \mathcal{D}'$ such that $L = H \circ \psi$.

Proof: Consider an arbitrary function $h \in \mathcal{A}(S)$. The function $\tilde{h}$ given as

$$\tilde{h}(\boldsymbol{z}) = h(\boldsymbol{z}) - App_{\boldsymbol{\beta}}(h)(\boldsymbol{z}), \quad \boldsymbol{z} \in S,$$

is an element of $\mathcal{A}(S)$, and for all $\boldsymbol{\gamma} \in \mathbb{N}^n$ with $\boldsymbol{\gamma} \leq \boldsymbol{\beta}$ we have

$$App_{\boldsymbol{\gamma}}(\tilde{h}) = App_{\boldsymbol{\gamma}}(h) - App_{\boldsymbol{\gamma}}(App_{\boldsymbol{\beta}}(h)) = 0,$$

so that $L(\tilde{h}) = 0$, i.e., $L(h) = L(App_{\boldsymbol{\beta}}(h))$, for every $h \in \mathcal{A}(S)$. Define the map H from $\mathcal{D}$ to $\mathbb{C}$ given by

$$H((f_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n}) = L(f_{\boldsymbol{\beta}}), \quad (f_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \mathcal{D}.$$

The natural injection from $\mathcal{D}$ into $\mathcal{P}$, the projection from $\mathcal{P}$ to $\mathcal{A}_{\boldsymbol{\beta}}$, and the natural injection from $\mathcal{A}_{\boldsymbol{\beta}}$ into $\mathcal{A}(S)$, are continuous mappings; hence $H \in \mathcal{D}'$. On the other hand, for $f \in \mathcal{A}(S)$,

$$(H \circ \psi)(f) = H((App_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}) = L(App_{\boldsymbol{\beta}}(f)) = L(f),$$

as desired. □

The last three propositions allow us to deduce that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{A}(S)'$, and hence, the surjectivity of ψ. æ

5 A new Borel-Ritt interpolation problem.

In this section we shall prove the corresponding Borel-Ritt theorem in the space $\mathcal{B}(S)$ of holomorphic functions on an unbounded polysector S of $\mathbb{C}^n$ whose derivatives are bounded on the unbounded proper subpolysectors of S. Section 3 may be repeated word by word for $\mathcal{B}(S)$, just changing bounded proper subpolysectors of S into unbounded ones. So, elements of $\mathcal{B}(S)$ are strongly asymptotically developable at $\mathbf{0}$ in S in a somewhat different sense; having this in mind, we use the same phrasing in this section.

Borel-Ritt interpolation problem is stated as follows:

Given a coherent family $\mathcal{F} = \{ f_{\boldsymbol{\alpha}_J} \in \mathcal{B}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J \}$, *does there exist* $f \in \mathcal{B}(S)$ *such that* $TA(f) = \mathcal{F}$*?*

The main difference between the settings in $\mathcal{A}(S)$ and $\mathcal{B}(S)$ is that, whereas in the first one the approximate functions to an element do remain in the considered space, in the second one this need not to be so (see the situation in dimension one, in which approximate functions are polynomials). This makes it impossible to use the second approach (see Section 4) to solve the present problem. Instead, we shall adopt the first one, based on coherent first order families.

Define, for each $j \in N$ and $m \in \mathbb{N}$, the space $\mathcal{B}_{jm} = \mathcal{B}(S_{j^c})$, provided with the natural topology. $\mathcal{B}_{jm}$ is a Fréchet space. Denote by $\mathcal{P}$ the product space $\prod_{\substack{j\in N\\ m\in\mathbb{N}}} \mathcal{B}_{jm}$, endowed with the product topology, and let $\mathcal{D}$ be the subspace of $\mathcal{P}$ consisting of coherent first order families $\mathcal{F}' = \{ f_{jm} \}$. The subspace topology makes $\mathcal{D}$ a Fréchet space. The map ψ which sends a function $f \in \mathcal{B}(S)$ to its first order family $TA'(f) \in \mathcal{D}$ is continuous, since so they are the maps ψ_{jm}, $j \in N$, $m \in \mathbb{N}$, sending $f \in \mathcal{B}(S)$ to $f_{jm} \in \mathcal{B}_{jm}$.

Our aim is to prove that ψ is surjective; we will apply the same argument as in the previous section. Due to the way $\mathcal{D}$ and its topology have been defined, $\psi(\mathcal{B}(S))$ is dense in $\mathcal{D}$ if we prove the possibility of interpolating finitely many elements of a family in $\mathcal{D}$ (in fact, in a "continuous" way, what will be decisive later on). This will be done in Proposition 5.3, after two auxiliary results. The first one is the one-dimensional Borel-Ritt theorem in this context.

Theorem 5.1 *Let S be an unbounded sector of $\mathbb{C}$ with vertex at the origin. For any sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, there exists a holomorphic function f from S to $\mathbb{C}$ such that for every proper subsector T of S,*

$$\sup_{z\in T}\left|\frac{f(z)-\sum_{j=0}^{m-1}a_jz^j}{z^m}\right|<+\infty,\quad m\in\mathbb{N}.$$

In this situation, we will adopt the notation $f\sim_S\sum_{n=0}^{\infty}a_nz^n$.
Proof: It is easy to reduce the problem to the sector $S=\{z:|\arg(z)|<\frac{\pi}{4}\}$. By the classical Borel's theorem, we can construct a complex function q in $\mathcal{C}^\infty(\mathbb{R})$, with compact support, and such that $q^{n)}(0)=a_n$, $n\in\mathbb{N}$. Then, the function

$$F(z)=\int_0^\infty e^{-tz}q(t)\,dt,\qquad z\in S,$$

is holomorphic in S, and $f(z)=z^{-1}F(z^{-1})$ solves the problem in S, as it can be seen by taking $\delta=\pi/4$ and $\sigma=0$ in [Ol, Chapter 4, §1.1, p. 106].□

Lemma 5.2 *Let $n\in\mathbb{N}$, $n>1$, and $N=\{1,2,\ldots,n\}$. Suppose J is a nonempty proper subset of N, and consider functions $f\in\mathcal{B}(S_{J^c})$ and $g\in\mathcal{B}(S_J)$ with $TA'(f)=\{f_{jm}\}$ and $TA'(g)=\{g_{jm}\}$. Then, $fg\in\mathcal{B}(S)$ and, if $TA'(fg)=\{h_{jm}\}$, it holds*

$$h_{jm}(\boldsymbol{z}_{j^c})=\begin{cases} f_{jm}(\boldsymbol{z}_{(J\cup\{j\})^c})g(\boldsymbol{z}_J), & \text{if } j\in J^c;\\ g_{jm}(\boldsymbol{z}_{J-\{j\}})f(\boldsymbol{z}_{J^c}), & \text{if } j\in J.\end{cases}$$

Proposition 5.3 *Let S be a polysector of $\mathbb{C}^n$ with vertex at $\mathbf{0}$. The following statements hold:*

i) Given $\mathcal{F}'=\{f_{jm}\}\in\mathcal{D}$ and $p\in\mathbb{N}$, there exists $F\in\mathcal{B}(S)$ such that, if $TA'(F)=\{F_{jm}\}$, then $F_{jm}=f_{jm}$, $j=1,2,\ldots,n$, $m=0,1,\ldots,p$.

ii) Let $\{\mathcal{F}'_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{D}$ converging to 0, where $\mathcal{F}'_k=\{f_{jm,k}\}$, $k\in\mathbb{N}$. Given $p\in\mathbb{N}$, there exists a sequence $\{F_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}(S)$

converging to 0*, such that, if* $TA'(F_k) = \{\, F_{jm,k}\,\}$*,* $k \in \mathbb{N}$*, then* $F_{jm,k} = f_{jm,k}$*,* $j = 1, 2, \dots, n$*,* $m = 0, 1, \dots, p$*.*

For the sake of a better understanding of the procedure, and trying to get rid of a cumbersome notation, we will limit ourselves to the case $n = 2$. The proof for an arbitrary dimension is analogous.

In the two-dimensional case the statement is as follows: Let $S = S_1 \times S_2$ be a polysector of $\mathbb{C}^2$ with vertex at the origin.

i) Consider a coherent first order family $\mathcal{F}' = \{\, f_n \in \mathcal{B}(S_1),\ g_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\,\}$, i.e., such that for every $m, n \in \mathbb{N}$, and for every proper subsectors T_1 of S_1 and T_2 of S_2, we have

$$\lim_{\substack{z\to 0\\ z\in T_1}} \frac{f_n^{m)}(z)}{m!} = \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{g_m^{n)}(\omega)}{n!}. \tag{4}$$

Then, given $p \in \mathbb{N}$, there exists $F \in \mathcal{B}(S)$ with

$$TA'(F) = \{\, h_n \in \mathcal{B}(S_1),\ \ell_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\,\},$$

verifying

$$h_n = f_n \quad \text{and} \quad \ell_m = g_m, \quad n, m = 0, 1, \dots, p.$$

ii) Consider, for $k \in \mathbb{N}$, coherent first order families

$$\mathcal{F}'_k = \{\, f_{n,k} \in \mathcal{B}(S_1),\ g_{m,k} \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\,\},$$

such that $\{\mathcal{F}'_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{D}$.

Then, given $p \in \mathbb{N}$, there exists a sequence $\{F_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}(S)$, that converges to 0, such that, if

$$TA'(F_k) = \{\, h_{n,k} \in \mathcal{B}(S_1),\ \ell_{m,k} \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\,\}, \quad k \in \mathbb{N},$$

we have $h_{n,k} = f_{n,k}$ and $\ell_{m,k} = g_{m,k}$, $n, m = 0, 1, \dots, p$, $k \in \mathbb{N}$.
Proof: for the sake of brevity, we write $\{f_n, g_m\}$ instead of

$$\{\, f_n \in \mathcal{B}(S_1),\ g_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\,\}.$$

The proof of item i) will be carried out in two steps. In the following, $T = T_1 \times T_2$ denotes an arbitrary proper subpolysector of S.

Step 1.- We will obtain a function $F \in \mathcal{B}(S)$ such that, if $TA'(F) = \{ h_n,\ \ell_m \}$, then $h_n = f_n$, $n = 0, 1, \ldots, p$, and $\ell_0 = g_0$. Indeed, in virtue of Theorem 5.1, we can consider, for $j = 0, 1, \ldots, p$, a function $\alpha_j \in \mathcal{B}(S_2)$ such that $\alpha_j(\omega) \sim_{S_2} \omega^j$. Let us define the function G_j from S to $\mathbb{C}$ as $G_j(z, \omega) = f_j(z)\alpha_j(\omega)$, $(z, \omega) \in S$. According to Lemma 5.2, $G_j \in \mathcal{B}(S)$. Let $TA'(G_j) = \{ h_{n,j},\ \ell_{m,j} \}$. By the choice of α_j, for $z \in S_1$, we have

$$h_{n,j}(z) = \lim_{\substack{\omega \to 0 \\ \omega \in T_2}} \frac{D^{(0,n)} G_j(z, \omega)}{n!} = f_j(z)\delta_{n,j},$$

where $\delta_{n,j}$ equals 1 if $n = j$ and 0 if $n \neq j$. Hence, $h_{n,j} = 0$ if $n \neq j$, and $h_{j,j} = f_j$. The function $G = \sum_{i=1}^{p} G_i$ belongs to $\mathcal{B}(S)$, and, if $TA'(G) = \{ H_n,\ L_m \}$, it is clear that $H_n = 0$ for $n > p$, and $H_n = f_n$ for $n \leq p$.

Let $\beta_0 \in \mathcal{B}(S_1)$ be such that $\beta_0(z) \sim_{S_1} 1$, and define the function M from S to $\mathbb{C}$ by

$$M(z, \omega) = \beta_0(z)(g_0(\omega) - L_0(\omega)), \quad (z, \omega) \in S.$$

By Lemma 5.2, $M \in \mathcal{B}(S)$. If $TA'(M) = \{ \tilde{H}_n,\ \tilde{L}_m \}$, then, according to (4) and to the coherence conditions for the family $TA'(G)$, for $n \leq p$ and $z \in S_1$,

$$\tilde{H}_n(z) = \lim_{\substack{\omega \to 0 \\ \omega \in T_2}} \frac{D^{(0,n)} M(z, \omega)}{n!} = \beta_0(z) \lim_{\substack{\omega \to 0 \\ \omega \in T_2}} \frac{g_0^{n)}(\omega) - L_0^{n)}(\omega)}{n!}$$
$$= \beta_0(z) \lim_{\substack{z \to 0 \\ z \in T_1}} (f_n(z) - H_n(z)) = 0;$$

on the other hand, by the choice of β_0, for every $\omega \in S_2$ we have

$$\tilde{L}_0(\omega) = \lim_{\substack{z \to 0 \\ z \in T_1}} M(z, \omega) = (g_0(\omega) - L_0(\omega)) \lim_{\substack{z \to 0 \\ z \in T_1}} \beta_0(z) = g_0(\omega) - L_0(\omega).$$

The additivity of first order families allows us to conclude that the function $F = G + M \in \mathcal{B}(S)$ is a solution for the first step.

Step 2.- To get the mentioned result, we will use recurrence, assuming that there exists $G \in \mathcal{B}(S)$ such that, if $TA'(G) = \{ H_n,\ L_m \}$, then $H_n = f_n$ if $n \leq p$ and $L_m = g_m$ if $m \leq p - 1$.

Consider a function $\beta_p \in \mathcal{B}(S_1)$ such that $\beta_p(z) \sim_{S_1} z^p$, and define the function M from S to $\mathbb{C}$ as

$$M(z, \omega) = \beta_p(z)(g_p(\omega) - L_p(\omega)), \quad (z, \omega) \in S.$$

Applying again Lemma 5.2, $M \in \mathcal{B}(S)$; say $TA'(M) = \{ \tilde{H}_n,\ \tilde{L}_m \}$. Because of (4) and of the coherence conditions for $TA'(G)$, for $n \leq p$ and $z \in S_1$,

$$\tilde{H}_n(z) = \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{D^{(0,n)}M(z,\omega)}{n!} = \beta_p(z) \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{g_0^{n)}(\omega) - L_0^{n)}(\omega)}{n!}$$

$$= \beta_p(z) \lim_{\substack{z\to 0\\ z\in T_1}} (f_n(z) - H_n(z)) = 0,$$

whereas, from the choice of β_p, for $m \leq p$ and $\omega \in S_2$ we have

$$\tilde{L}_m(\omega) = \lim_{\substack{z\to 0\\ z\in T_1}} \frac{D^{(m,0)}M(z,\omega)}{m!}$$

$$= (g_p(\omega) - L_p(\omega)) \lim_{\substack{z\to 0\\ z\in T_1}} \frac{\beta_p^{m)}(z)}{m!} = (g_p(\omega) - L_p(\omega))\delta_{mp}.$$

The function $F = G + M \in \mathcal{B}(S)$ is the solution we were looking for.

ii) We will again divide the proof in two steps.

Step 1.- Our aim is to show the existence of a sequence $\{F_k\}_{k\in\mathbb{N}}$ of functions of $\mathcal{B}(S)$, converging to 0, and such that, if $TA'(F_k) = \{h_{n,k}, \ell_{m,k}\}$, $k \in \mathbb{N}$, then $h_{n,k} = f_{n,k}$, $n = 0, 1, \ldots, p$, and $\ell_{0,k} = g_{0,k}$.

For $j = 0, 1, \ldots, p$, consider a function $\alpha_j \in \mathcal{B}(S_2)$ such that $\alpha_j(\omega) \sim_{S_2} \omega^j$, and define the element $G_{j,k}$ of $\mathcal{B}(S)$ as $G_{j,k}(z,\omega) = f_{j,k}(z)\alpha_j(\omega)$, $(z,\omega) \in S$. We claim that $\{G_{j,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$, for $j = 0, 1, \ldots, p$: for every proper subpolysector $T = T_1 \times T_2$ of S and every multiindex $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{N}^2$,

$$Q_{T,\boldsymbol{\gamma}}(G_{j,k}) = \sup_{(z,\omega)\in T} |D^{\boldsymbol{\gamma}} G_{j,k}(z,\omega)| \leq \sup_{z\in T_1} |f_{j,k}^{\gamma_1)}(z)| \sup_{\omega\in T_2} |\alpha_j^{\gamma_2)}(\omega)|$$

$$= C(T_2, j, \gamma_2) Q_{T_1,\gamma_1}(f_{j,k}).$$

As $\{f_{j,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S_1)$, the conclusion is immediate.

For $k \in \mathbb{N}$, $G_k = \sum_{i=1}^{p} G_{i,k} \in \mathcal{B}(S)$; say $TA'(G_k) = \{H_{n,k}, L_{m,k}\}$. The sequence $\{G_k\}_{k\in\mathbb{N}}$ obviously converges to 0 in $\mathcal{B}(S)$. This fact and the continuity of the map that sends each element of $\mathcal{B}(S)$ to the corresponding element of its first order family allow us to deduce the convergence to 0 of the sequence $\{L_{0,k}\}_{k\in\mathbb{N}}$ in $\mathcal{B}(S_2)$.

Let $\beta_0 \in \mathcal{B}(S_1)$ be such that $\beta_0(z) \sim_{S_1} 1$, and define, for $k \in \mathbb{N}$, the function M_k of $\mathcal{B}(S)$ given by

$$M_k(z,\omega) = \beta_0(z)(g_{0,k}(\omega) - L_{0,k}(\omega)), \quad (z,\omega) \in S.$$

We have

$$Q_{T,\boldsymbol{\gamma}}(M_k) = \sup_{(z,\omega)\in T} |D^{\boldsymbol{\gamma}} M_k(z,\omega)| \leq \sup_{z\in T_1} |\beta_0^{\gamma_1)}(z)| \sup_{\omega\in T_2} |(g_{0,k} - L_{0,k})^{\gamma_2)}(\omega)|$$

$$\leq C(T_1,\gamma_1)(Q_{T_2,\gamma_2}(g_{0,k}) + Q_{T_2,\gamma_2}(L_{0,k})).$$

As $\{g_{0,k}\}_{k\in\mathbb{N}}$ and $\{L_{0,k}\}_{k\in\mathbb{N}}$ converge to 0 in $\mathcal{B}(S_2)$, also $\{M_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$. So, the sequence $\{F_k\}_{k\in\mathbb{N}}$ defined as $F_k = G_k + M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, converges to 0 in $\mathcal{B}(S)$. The proof of the first step of item i) shows that the rest of the statement holds.

Step 2.- We will use an induction argument. Assume therefore that there exists a sequence $\{G_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{B}(S)$ that converges to 0 and such that, if for $k \in \mathbb{N}$, $TA'(G_k) = \{H_{n,k}, L_{m,k}\}$, then $H_{n,k} = f_{n,k}$ if $n \leq p$ and $L_{m,k} = g_{m,k}$ if $m \leq p-1$. Notice that the sequence $\{L_{p,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S_2)$.

Consider a function $\beta_p \in \mathcal{B}(S_1)$ such that $\beta_p(z) \sim_{S_1} z^p$, and define, for $k \in \mathbb{N}$, the function M_k in $\mathcal{B}(S)$ as

$$M_k(z,\omega) = \beta_p(z)(g_{p,k}(\omega) - L_{p,k}(\omega)), \quad (z,\omega) \in S.$$

We have

$$Q_{T,\boldsymbol{\gamma}}(M_k) = \sup_{(z,\omega)\in T} |D^{\boldsymbol{\gamma}} M_k(z,\omega)| \leq \sup_{z\in T_1} |\beta_p^{\gamma_1)}(z)| \sup_{\omega\in T_2} |(g_{p,k} - L_{p,k})^{\gamma_2)}(\omega)|$$

$$\leq C(T_1,\gamma_1)(Q_{T_2,\gamma_2}(g_{p,k}) + Q_{T_2,\gamma_2}(L_{p,k})).$$

Since the sequences $\{g_{p,k}\}_{k\in\mathbb{N}}$ and $\{L_{p,k}\}_{k\in\mathbb{N}}$ converge to 0 in $\mathcal{B}(S_2)$, we deduce that $\{M_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$. Hence the sequence $\{F_k\}_{k\in\mathbb{N}}$, defined by $F_k = G_k + M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, converges to 0 in $\mathcal{B}(S)$, and the proof of the second step in i) leads to the desired result. □

Our last task consists in proving that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{B}(S)'$. To this end, we will need the following three propositions. The proof for the first one resembles that of Proposition 4.1, just changing the space $\mathcal{A}(S)$ into $\mathcal{B}(S)$ and the spaces $\mathcal{A}_{\boldsymbol{\alpha}}$ into the spaces $\mathcal{B}_{jm}$.

Proposition 5.4 *Let L be a continuous functional on $\mathcal{B}(S)$ that belongs to the weak closure of ${}^t\psi(\mathcal{D}')$. Then,*

$$\bigcap_{j\in N, m\in\mathbb{N}} \mathrm{Ker}(\psi_{jm}) \subset \mathrm{Ker}(L),$$

i.e., for every $f \in \mathcal{B}(S)$ such that $\psi_{jm}(f) = f_{jm} = 0$ for all $j \in N$ and for all $m \in \mathbb{N}$, $L(f) = 0$ holds.

The next lemma is deduced from the coherence conditions for the total family associated to an element of $\mathcal{B}(S)$.

Lemma 5.5 *Let $f \in \mathcal{B}(S)$ and $\boldsymbol{\alpha} \in \mathbb{N}^n$. Then, $App_{\boldsymbol{\alpha}}(f) = 0$ if and only if for every $j \in N$ and $m \in \mathbb{N}$ such that $\alpha_j \neq 0$ and $m \leq \alpha_j - 1$, we have $f_{jm} = 0$.*

Proposition 5.6 *Let L be an element of $\mathcal{B}(S)'$ that satisfies the following property:*

If $f \in \mathcal{B}(S)$ is such that $f_{jm} = 0$ for every $j \in N$ and every $m \in \mathbb{N}$, then $L(f) = 0$.
Then, there exists $r \in \mathbb{N}$, $r \geq 1$, such that for every $f \in \mathcal{B}(S)$ satisfying $f_{jm} = 0$ for $j \in N$ and $m \in \mathbb{N}$ with $m \leq r$, we have $L(f) = 0$.

Proof: Let $g \in \mathcal{B}(S)$ be such that $g_{jm} = 0$ for $j \in N$ and $m \in \mathbb{N}$ with $m \leq r$. By the previous lemma, this implies that $App_{\boldsymbol{\alpha}}(g) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $0 \leq |\boldsymbol{\alpha}| \leq r + 1$. Therefore, the proof follows as in Proposition 4.2. □

Proposition 5.7 *Let $L \in \mathcal{B}(S)'$ such that there exists $r \in \mathbb{N}$, $r \geq 1$, satisfying the following property:*

If $f \in \mathcal{B}(S)$ and $f_{jm} = 0$ for every $j \in N$ and every $m \in \mathbb{N}$ with $m \leq r$, then $L(f) = 0$.
Then, there exists a functional $H \in \mathcal{D}'$ such that $L = H \circ \psi$.

Proof: According to the proof of Proposition 5.3, for all $\mathcal{G}' = \{ g_{jm} \} \in \mathcal{D}$ there exists a function $G \in \mathcal{B}(S)$, with $TA'(G) = \{ G_{jm} \}$, such that for $j \in N$ and $m \in \mathbb{N}$, $m \leq r$, we have $G_{jm} = g_{jm}$.

Define the map H from $\mathcal{D}$ to $\mathbb{C}$ given by $H(\mathcal{G}') = L(G)$, $\mathcal{G}' \in \mathcal{D}$. H is well defined, as it can be easily deduced from the hypothesis imposed on L. For the same reason, it is clear that $L = H \circ \psi$.

To prove the continuity of H, consider a sequence $\{\mathcal{G}'_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{D}$ converging to 0, where $\mathcal{G}'_k = \{\, g_{jm,k} \,\}$, $k \in \mathbb{N}$. This implies the convergence to 0 in the corresponding spaces $\mathcal{B}_{jm}$ of the sequences $\{g_{jm,k}\}_{k\in\mathbb{N}}$. As it was shown in Proposition 5.3, there exists a sequence $\{G_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{B}(S)$ converging to 0, and such that, if $TA'(G_k) = \{\, G_{jm,k} \,\}$, $k \in \mathbb{N}$, then, for $j \in N$ and for $m \in \mathbb{N}$ with $m \leq r$, we have $G_{jm,k} = g_{jm,k}$. So, $H(\mathcal{G}'_k) = L(G_k)$, $k \in \mathbb{N}$. Now, the continuity of L implies $\lim_{k\to\infty} L(G_k) = 0$, and so, H is continuous. Its linearity results immediately. $\square$

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