Summability in a direction of formal power series in several variables

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Abstract

In this paper a theory of summability in a direction, extending that of J. P. Ramis, is developed for formal power series of several variables. To this end, generalized Laplace and Borel transforms are studied, as well as their action on functions admitting Gevrey strongly asymptotic expansion as defined by H. Majima. The definition we give of summability in a direction turns out to be, in a sense, equivalent to an iterative classical summation procedure. As an application we provide a new proof of a well-known result of R. Gérard and Y. Sibuya stating the convergence of the formal power series solution to a certain completely integrable Pfaffian system.

1 Introduction

The theory of k-summability ($k > 0$) of formal power series in one variable was developed by J. P. Ramis [18, 19] in the seventies. He proved, by a purely theoretical (not explicit) method, that every formal solution to a linear system of meromorphic ordinary differential equations in the complex domain at an irregular singular point can be written as some known functions times a finite product of power series, each of which is k-summable for some k depending on the series. The non-constructive character of the proof was solved by the introduction of a more powerful tool, multisummability, due to J. Ecalle [6, 7]. Indeed, it has been shown [5, 20, 1] that every formal power series solution to nonlinear systems of ODE's is multisummable, which allows to compute actual solutions from formal ones.

Regarding the study of formal power series solutions of partial differential equations, some work has been done in several directions. Different concepts of asymptotic expansion in several variables have been considered, such as those given by R. Gérard and Y. Sibuya [8], and H. Majima [14, 15]. In both cases, existence of formal solutions and of analytic solutions whose asymptotic expansions are given by the previous ones has been proved for certain classes of (Pfaffian) systems of PDE's, but no method is provided to construct those actual solutions. Also, in [9, 17] Gevrey type bounds (that is, those necessary for summability) have been obtained for the coefficients of formal power series solutions to Cauchy problems for certain PDE's. Concerning (multi)summability, no definition for series in several variables is known to us; however, summability theory has been applied in such cases (see the works of D. A. Lutz, M. Miyake and R. Schäfke [13], W. Balser [3] and W. Balser and M. Miyake [4]) by considering all but one of the variables as parameters, and then working with a series in one variable whose coefficients are functions, elements of some suitably chosen Banach space.

After the previous comments, it seems desirable to find a summation method that treats all variables equally and at the same time, as it has been explicitly pointed out by authors such as Y. Sibuya [26] and W. Balser [2, Chapter 13]. In this paper we try to do so. Our definition of summability in a direction for series of several variables relies on the concept of strongly asymptotically developable functions of H. Majima, that shares the usual stability properties (in particular, with respect to differentiation) of H. Poincaré's definition in one variable. In order to gain in clarity of notation and exposition, most of the paper deals with series and functions in two variables, though all the results remain valid for any number of variables, as briefly indicated in the last Section.

After some notations (Section 2), the basic facts on Gevrey series and (Gevrey) strongly asymptotically developable functions on polysectors are recalled in Section 3. In case the polysector is "wide", Proposition 3.3 (or Proposition 7.1 for any number of variables) establishes the injectivity of the mapping sending a Gevrey function to its formal series of asymptotic expansion. This property, the analogue to Watson's lemma in one variable, is fundamental for the definition of summability. Also in this Section, Gevrey functions with null asymptotics are characterized in terms of a certain exponential decrease, and a definition of function of exponential growth in an unbounded polysector is given.

Section 4 provides the definition of multidimensional Laplace and Borel transforms, and the results on how they both act on Gevrey functions (with a given exponential growth in the first case). These transforms are also proved to be the inverse of each other.

Our definition of $\boldsymbol{k}$-summability in a direction $\boldsymbol{d}$ is given in Section 5, resembling the definition for one variable. In a natural way the question appears of whether the "double" sum of a summable series in two variables can be obtained by iterating the one-variable summation procedure. The answer is affirmative (Proposition 5.4), due to the isomorphism established in Proposition 5.2, and to the fact that summability theory is valid for one-variable series (resp. functions) with coefficients (resp. values) in a Banach space (see [2, Chapters 4–6]). Some basic properties are stated, and special attention is paid (Remark 5.7) to the case where convergence may be guaranteed in some of the steps of summation. Subsequently, a definition of $\boldsymbol{k}$-summability is proposed.

A thorough study of the properties deriving from this last definition should be made. Other topics in the one-variable theory, such as Cauchy-Heine transforms, decomposition theorems, multisummability, J. Ecalle's acceleration operators, etc. seem to be amenable to generalization. The next task would be to prove (multi)summability of the formal power series solutions of PDE's obtained in diverse instances (see above). As a first attempt, in Section 6 a new approach to a well-known result of R. Gérard and Y. Sibuya is considered. The formal power series solution to a certain class of completely integrable Pfaffian system can be shown to be (k_1,∞)-summable (resp. (∞,k_2)-summable) except in the directions d_1 (resp. d_2) in a finite set E_1 (resp. E_2). This implies convergence.

2 Notations

Denote by $\mathbb{R}$ the Riemann surface of the Logarithm, and one of its elements by $z=(r,\varphi)=re^{i\varphi}$. Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, $\boldsymbol{\beta}=(\beta_1,\beta_2)\in\mathbb{N}^2$, $m\in[0,\infty)$, $\boldsymbol{t}=(t_1,t_2)\in[0,\infty)^2$ and $\boldsymbol{z}=(z_1,z_2)\in\mathbb{R}^2$. We set

$$\begin{array}{ll}
\boldsymbol{\alpha}+\boldsymbol{\beta}=(\alpha_1+\beta_1,\alpha_2+\beta_2) & m\boldsymbol{t}=(mt_1,mt_2)\\
|\boldsymbol{\alpha}|=\alpha_1+\alpha_2 & \boldsymbol{\alpha}!=\alpha_1!\,\alpha_2!\\
\boldsymbol{\alpha}\le\boldsymbol{\beta}\Leftrightarrow\alpha_j\le\beta_j,\ j=1,2 & \boldsymbol{\alpha}<\boldsymbol{\beta}\Leftrightarrow\alpha_j<\beta_j,\ j=1,2\\
\mathbf{1}=(1,1) & \boldsymbol{e}_1=(1,0),\ \boldsymbol{e}_2=(0,1)\\
|\boldsymbol{z}^{\boldsymbol{\alpha}}|=|\boldsymbol{z}|^{\boldsymbol{\alpha}}=|z_1|^{\alpha_1}|z_2|^{\alpha_2} & \boldsymbol{z}^{\boldsymbol{\alpha}}=z_1^{\alpha_1}z_2^{\alpha_2}\\
D^{\boldsymbol{\alpha}}=\frac{\partial^{\boldsymbol{\alpha}}}{\partial\boldsymbol{z}^{\boldsymbol{\alpha}}}=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}} &
\end{array}$$

Consider, for $j=1,2$, an *open sector* (on the Riemann surface of the Logarithm) with vertex at the origin,

$$S_j=S(d_j,\theta_j,\rho_j)=\{\,z=re^{i\varphi}:0<r<\rho_j,\ |\varphi-d_j|<\theta_j/2\,\},$$

where $d_j\in\mathbb{R}$, $\theta_j>0$ and $\rho_j\in(0,\infty]$ are the *bisecting direction*, the *width* and the *radius* of S_j, respectively. The cartesian product $\prod_{j=1}^2 S_j=\prod_{j=1}^2 S(d_j,\theta_j,\rho_j)\subset\mathbb{R}^2$ of open sectors with vertex at 0 will be called a *polysector* with vertex at $\mathbf{0}$, and it will be denoted by $S=S(\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\rho})$, where $\boldsymbol{d}=(d_1,d_2)$, $\boldsymbol{\theta}=(\theta_1,\theta_2)$ and $\boldsymbol{\rho}=(\rho_1,\rho_2)$. In case $\rho_j=+\infty$ for $j=1,2$, we write $S=S(\boldsymbol{d},\boldsymbol{\theta})$.

We say a polysector $T=\prod_{j=1}^2 T(d_j',\theta_j',\rho_j')$ on $\mathbb{R}^2$ is a *bounded proper subpolysector* of $S=S(\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\rho})$, and we write $T\ll S$, if for $j=1,2$ we have $\rho_j'<\rho_j$ (so that $\rho_j'<+\infty$) and

$$[d_j'-\rho_j'/2,d_j'+\rho_j'/2]\subset(d_j-\rho_j/2,d_j+\rho_j/2). \tag{1}$$

Finally, we say $T=\prod_{j=1}^2 T(d_j',\theta_j')$ is an *unbounded proper subpolysector* of $S=S(\boldsymbol{d},\boldsymbol{\theta})$, and we write $T\prec S$, if for $j=1,2$ we have (1).

3 Definitions and fundamental properties

Let $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ be a polysector, and $\boldsymbol{k} = (k_1, k_2) \in (0, \infty)^2$.

A formal series $\widehat{f} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ with complex coefficients $a_{\boldsymbol{\alpha}}$ is said to be *Gevrey of order* $\boldsymbol{k}$ if there exist $C > 0$ and $A = (A_1, A_2) \in (0, \infty)^2$ such that for every $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ we have

$$|a_{\boldsymbol{\alpha}}| \le C\,\Gamma(1 + \alpha_1/k_1)\Gamma(1 + \alpha_2/k_2)\, A_1^{\alpha_1} A_2^{\alpha_2}.$$

As a shorthand for $\Gamma(1+\alpha_1/k_1)\Gamma(1+\alpha_2/k_2)$ we will write $\Gamma(1+\boldsymbol{\alpha}/\boldsymbol{k})$. We denote by $\mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$ the set of such series, which can be shown to be a differential algebra with respect to the standard operations: addition, multiplication with scalars, multiplication of power series and (formal) derivation. The set of all convergent power series (i.e., those converging in some polydisk) will be denoted by $\mathbb{C}\{\boldsymbol{z}\}$.

A holomorphic function $f\colon S \to \mathbb{C}$ is said to be

(i) *bounded at the origin* if it is bounded in every $T \ll S$;

(ii) *continuous at the origin* if $\lim_{\boldsymbol{z}\to 0,\ \boldsymbol{z}\in T} f(\boldsymbol{z})$ exists for every $T \ll S$;

(iii) *Gevrey of order* $\boldsymbol{k}$ if for every $T \ll S$ there exist $C_T > 0$ and $A_T \in (0, \infty)^2$ such that

$$\sup_{\boldsymbol{z}\in T} \frac{|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})|}{\boldsymbol{\alpha}!} \le C_T\,\Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A_T^{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathbb{N}^2;$$

$\mathcal{W}_{\boldsymbol{k}}(S)$ will be the set consisting of such functions, and it turns out to be a differential algebra with respect to the usual operations.

(iv) *Gevrey strongly asymptotically developable of order* $\boldsymbol{k}$ if there exists a family $\mathcal{F}(f) = \{ h_m, g_n, a_{nm} : n, m \in \mathbb{N} \}$, where h_m (resp. g_n) is a holomorphic function from S_1 (resp. S_2) to $\mathbb{C}$ and $a_{nm} \in \mathbb{C}$, $n, m \in \mathbb{N}$, such that the following holds: if we define for $\boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2$ the *approximate function*

$$\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z}) = \sum_{j=0}^{n-1} g_j(z_2) z_1^j + \sum_{\ell=0}^{m-1} h_\ell(z_1) z_2^\ell - \sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} a_{j\ell} z_1^j z_2^\ell, \quad \boldsymbol{z} \in S,$$

then for every $T \ll S$ there exist $C_T > 0$ and $A_T \in (0, \infty)^2$ such that for every $\boldsymbol{\alpha} \in \mathbb{N}^2$,

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})| \le C_T\,\Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A_T^{\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T.$$

The differential algebra consisting of these functions will be denoted by $\mathcal{A}_{\boldsymbol{k}}(S)$. In case we have

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})| \le C_{T,\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T,$$

for a suitable $C_{T,\boldsymbol{\alpha}} > 0$ arbitrarily depending on T and $\boldsymbol{\alpha}$, we say f is *strongly asymptotically developable* in S; $\mathcal{A}(S)$ will be the differential algebra of these functions.

The following facts are well-known, and they can be found in [10, 21] for the Gevrey case, and in [11, 22] for general asymptotically developable functions.

Let $f \in \mathcal{A}_{\boldsymbol{k}}(S)$. The elements of $\mathcal{F}(f)$ are given by

$$\begin{aligned}
h_m(z_1) &= \lim_{\substack{z_2\to 0\\ z_2\in T_2}} \frac{D^{(0,m)} f(z_1, z_2)}{m!}, \quad z_1 \in S_1,\ m \in \mathbb{N},\\
g_n(z_2) &= \lim_{\substack{z_1\to 0\\ z_1\in T_1}} \frac{D^{(n,0)} f(z_1, z_2)}{n!}, \quad z_2 \in S_2,\ n \in \mathbb{N}, \qquad (2)\\
a_{nm} &= \lim_{\substack{(z_1,z_2)\to \mathbf{0}\\ (z_1,z_2)\in T_1\times T_2}} \frac{D^{(n,m)} f(z_1, z_2)}{n!\,m!}, \quad (n, m) \in \mathbb{N}^2,
\end{aligned}$$

for any $T_j \ll S_j$, $j = 1, 2$, and the first (resp. second) limits hold uniformly on every $T_1 \ll S_1$ (resp. $T_2 \ll S_2$). Hence the so-called *total family of strongly asymptotic expansion* associated with f, $\mathcal{F}(f)$, is unique and will be denoted by $\mathrm{TA}(f)$, and accordingly $\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})$ will be from now on $\mathrm{App}_{\boldsymbol{\alpha}}(f)$. The (formal) *series of strongly asymptotic expansion* is $\mathrm{FA}(f) := \sum_{(n,m)\in\mathbb{N}^2} a_{nm} z_1^n z_2^m$ (for functions of one variable $\mathrm{FA}(f)$ is just the series of asymptotic expansion in the usual sense; also, for functions of several variables it agrees with the asymptotic expansion in the sense of R. Gérard and Y. Sibuya [8, 11]).

The elements of $\mathrm{TA}(f)$ are linked by the following *consistency conditions*:

$$\lim_{\substack{z_1\to 0\\ z_1\in T_1}} \frac{h_m^{n)}(z_1)}{n!} = \lim_{\substack{z_2\to 0\\ z_2\in T_2}} \frac{g_n^{m)}(z_2)}{m!} = a_{nm}, \quad (n,m)\in\mathbb{N}^2. \tag{3}$$

Hence it can be deduced that $h_m \in \mathcal{A}_{k_1}(S_1)$, $g_n \in \mathcal{A}_{k_2}(S_2)$, and

$$\mathrm{TA}(h_m) \equiv \mathrm{FA}(h_m) = \sum_{n=0}^{\infty} a_{nm} z_1^n, \qquad \mathrm{TA}(g_n) \equiv \mathrm{FA}(g_n) = \sum_{m=0}^{\infty} a_{nm} z_2^m.$$

Hereafter, we will say that a family

$$\mathcal{F} = \{\, h_m \in \mathcal{H}(S_1),\ g_n \in \mathcal{H}(S_2),\ a_{nm} \in \mathbb{C} \colon n, m \in \mathbb{N} \,\}$$

is *consistent* if it verifies (3).

Y. Haraoka [10] has proven that $\mathcal{W}_{\boldsymbol{k}}(S) = \mathcal{A}_{\boldsymbol{k}}(S)$. It is straightforward that whenever $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ one has $\mathrm{FA}(f) \in \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$, and the well defined mapping $\mathrm{FA}\colon \mathcal{A}_{\boldsymbol{k}}(S) \to \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ is a homomorphism of differential algebras. The question arises of whether, given $\widehat{f} \in \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$, there exists $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ such that $\mathrm{FA}(f) = \widehat{f}$. The answer is affirmative if we impose some restriction on the width of S. This is the result known as Borel-Ritt-Gevrey theorem, that, for functions of one variable, can be found for example in [16], [1, p. 17]. In the case of complex functions of several variables, a proof was given by Y. Haraoka.

Theorem 3.1 ([10, §2,Theorem 1.(1)])**.** *Let $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$, and $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ be a polysector with width $\boldsymbol{\theta} = (\theta_1, \theta_2)$ such that $\theta_j \le \pi/k_j$, $j = 1, 2$. Then, the mapping* FA *is surjective.*

A different interpolation problem may be posed in terms of consistent families $\mathcal{F}$, searching for a function f such that $\mathrm{TA}(f) = \mathcal{F}$. Some results in this direction have been given in [10, §2,Theorem 1.(2)] and [21].

In the one variable case, the mapping FA is not injective if and only if $\theta \le \pi/k$ (see, for example, [1, pp. 18 and 23]). This easily implies that when $\boldsymbol{\theta}, \boldsymbol{k}$ are such that $\theta_j \le \pi/k_j$ for some j, then FA is not injective. Regarding TA, the mapping sending $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ to $\mathrm{TA}(f)$, we have

Proposition 3.2 ([21])**.** *Let $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$, and $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ be a polysector with width $\boldsymbol{\theta} = (\theta_1, \theta_2)$ such that $\theta_j > \pi/k_j$ for some $j \in \{1, 2\}$. Then, the mapping* TA *is injective.*

Proof. Without loss of generality, suppose $\theta_1 > \pi/k_1$. Let us fix an arbitrary element $z_2 \in S_2$, and let us observe that the function $f(\cdot, z_2)$ admits the null series as k_1-Gevrey asymptotic expansion. From the injectivity of $\mathrm{TA} \equiv \mathrm{FA}$ on $\mathcal{A}_{k_1}(S_1)$, we conclude that this function is identically zero. □

As a consequence, we obtain

Proposition 3.3. *Let $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$, and $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ be a polysector with width $\boldsymbol{\theta} = (\theta_1, \theta_2)$ such that $\theta_j > \pi/k_j$ for $j = 1, 2$. Then, the mapping $\mathrm{FA}\colon \mathcal{A}_{\boldsymbol{k}}(S) \to \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ is injective.*

Proof. Let $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ such that $\text{FA}(f) = \widehat{0}$ (the null series). For every $m \in \mathbb{N}$, the function h_m, element of $\text{TA}(f)$, belongs to $\mathcal{A}_{k_1}(S_1)$, and

$$\text{TA}(h_m) \equiv \text{FA}(h_m) = \sum_{n=0}^{\infty} a_{nm} z_1^n = \widehat{0}.$$

Thus, being $\theta_1 > \pi/k_1$, the injectivity of FA in the one variable case implies $h_m \equiv 0$ in S_1 for every $m \in \mathbb{N}$. In a similar fashion one can obtain that also $g_n \equiv 0$ in S_2 for every $n \in \mathbb{N}$. We conclude that $\text{TA}(f)$ is the null family, and the previous result implies $f \equiv 0$. □

In fact, and for the sake of completeness, one can determine precise conditions that amount to $\text{TA}(f)$ being the null family.

Proposition 3.4. *Let $f\colon S \to \mathbb{C}$ be holomorphic. The following are equivalent:*
(i) *$f \in \mathcal{A}_{\boldsymbol{k}}(S)$ and all the elements of* $\text{TA}(f)$ *are null.*
(ii) *For every $T \ll S$, there exist constants $C_T > 0$ and $M_T > 0$ such that for every $\boldsymbol{z} = (z_1, z_2) \in T$ one has*

$$|f(\boldsymbol{z})| \le C_T \exp\big(- M_T(|z_1|^{-k_1} + |z_2|^{-k_2})\big).$$

Proof. (i)⇒(ii) Given $T \ll S$, there exist $C_T > 0$ and $A_T = (A_1, A_2) \in (0,\infty)^2$ such that for every $\boldsymbol{\alpha} \in \mathbb{N}^2$ one has

$$|f(\boldsymbol{z})| = |f(\boldsymbol{z}) - \text{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})| \le C_T \,\Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A_T^{\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T.$$

In particular, for $n, m \in \mathbb{N}$ we can choose $\boldsymbol{\alpha} = (n, 0)$ or $\boldsymbol{\alpha} = (0, m)$ and write, respectively, for $\boldsymbol{z} \in T$,

$$|f(\boldsymbol{z})| \le C_T \,\Gamma(1 + n/k_1) A_1^n |z_1|^n, \qquad |f(\boldsymbol{z})| \le C_T \,\Gamma(1 + m/k_2) A_2^m |z_2|^m.$$

From Stirling's formula we obtain suitable $D_T > 0$ and $B_j > 0$ such that

$$|f(\boldsymbol{z})| \le D_T (B_1 |z_1|)^n n^{n/k_1}, \quad |f(\boldsymbol{z})| \le D_T (B_2 |z_2|)^m m^{m/k_2}, \qquad \boldsymbol{z} \in T.$$

For $r > 0$ and $j = 1, 2$, the function $g_j(x) = (B_j r)^x \, x^{x/k_j}$ reaches its absolute minimum in $(0,\infty)$ at $x_{0j} = e^{-1}(B_j r)^{-k_j}$. There exists $r_{0j} > 0$ such that whenever $r < r_{0j}$ we have $x_{0j} = x_{0j}(r) > 1$. Therefore, between x_{0j} and $2x_{0j}$ we can find an integer, and for every $\boldsymbol{z} = (z_1, z_2) \in T$ such that $J(\boldsymbol{z}) = \{j \colon |z_j| < r_{0j}\}$ is not the empty set we get

$$|f(\boldsymbol{z})| \le D_T g_j(2x_{0j}) = D_T \exp\Big[- \frac{2\log(e/2) B_j^{-k_j}}{k_j e} |z_j|^{-k_j} \Big], \qquad j \in J(\boldsymbol{z}).$$

The conclusion easily follows.
(ii)⇒(i) Fix $T \ll S$. For suitable C_T, $M_T > 0$,

$$|f(\boldsymbol{z})| \le C_T \exp\big[- M_T(|z_1|^{-k_1} + |z_2|^{-k_2})\big], \quad \boldsymbol{z} \in T.$$

Then, for every $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ we have

$$|f(\boldsymbol{z})/\boldsymbol{z}^{\boldsymbol{\alpha}}| \le C_T \prod_{j=1}^{2} (|z_j|^{-\alpha_j} \exp(-M_T |z_j|^{-k_j})), \quad \boldsymbol{z} \in T.$$

The absolute maximum of the function $g(x) = x^{-\alpha} \exp(-M x^{-k})$ in $(0,\infty)$ equals $\big(\frac{\alpha}{eMk}\big)^{\alpha/k}$, so that (again by Stirling's formula) we find $E_T > 0$ and $A_T = (A_1, A_2) \in (0,\infty)^2$ such that

$$|f(\boldsymbol{z})/\boldsymbol{z}^{\boldsymbol{\alpha}}| \le E_T \prod_{j=1}^{2} \big(\Gamma(1 + \alpha_j/k_j) A_j^{\alpha_j}\big) = E_T \,\Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A_T^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T,$$

as desired. □

Definition 3.5. A holomorphic function $f\colon S = S(\boldsymbol{d},\boldsymbol{\theta}) \to \mathbb{C}$ is *of exponential growth at most* $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$ *in* S if for every (unbounded and proper subpolysector) $T \prec S$, there exist $r > 0$, $C > 0$ and $M > 0$ (depending on T) such that whenever $\boldsymbol{z} = (z_1, z_2) \in T$ and $\|\boldsymbol{z}\|_\infty := \max\{|z_1|, |z_2|\} \geq r$,

$$|f(\boldsymbol{z})| \leq C \exp(M (|z_1|^{k_1} + |z_2|^{k_2})). \tag{4}$$

We say that a family of holomorphic functions $\{f_\beta\}_{\beta\in B}$ (for some set of indices B) is *uniformly of exponential growth at most* $\boldsymbol{k}$ *in* S if for every $T \prec S$, there exist $r = r(T) > 0$ and $M = M(T) > 0$, and for every $\beta \in B$ there exists $C_\beta = C_\beta(T) > 0$ such that for every $\boldsymbol{z} = (z_1, z_2) \in T$ with $\|\boldsymbol{z}\|_\infty \geq r$ we have

$$|f_\beta(\boldsymbol{z})| \leq C_\beta \exp(M (|z_1|^{k_1} + |z_2|^{k_2})).$$

4 Laplace and Borel transforms for Gevrey functions of several complex variables

For $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$, we denote by $(0,\infty)^2(\boldsymbol{t})$ the set

$$\{\boldsymbol{z} = (z_1, z_2) \in \mathcal{R}^2 \colon (z_1 e^{-it_1}, z_2 e^{-it_2}) \in (0,\infty)^2\}.$$

Let f be an analytic function defined in a polysector $S = S(\boldsymbol{d},\boldsymbol{\theta})$. Suppose f is of exponential growth at most $\boldsymbol{k} \in (0,\infty)^2$ in S, and f is also bounded at the origin. Then, for $\boldsymbol{t} = (t_1, t_2) \in (\boldsymbol{d}-\boldsymbol{\theta}/2, \boldsymbol{d}+\boldsymbol{\theta}/2)$ we define the *Laplace transform of* f *with index* $\boldsymbol{k}$ *in direction* $\boldsymbol{t}$, denoted by $\mathcal{L}_{\boldsymbol{k},\boldsymbol{t}} f$, as

$$(\mathcal{L}_{\boldsymbol{k},\boldsymbol{t}} f)(\boldsymbol{z}) = \frac{1}{\boldsymbol{z}^{\boldsymbol{k}}} \int_{(0,\infty)^2(\boldsymbol{t})} f(u_1, u_2) \prod_{j=1}^{2} \left[e^{-(u_j/z_j)^{k_j}} k_j u_j^{k_j-1}\right] du_1 du_2.$$

As a shorthand, we will write

$$(\mathcal{L}_{\boldsymbol{k},\boldsymbol{t}} f)(\boldsymbol{z}) = \frac{1}{\boldsymbol{z}^{\boldsymbol{k}}} \int_{(0,\infty)^2(\boldsymbol{t})} f(\boldsymbol{u}) e^{-(\boldsymbol{u}/\boldsymbol{z})^{\boldsymbol{k}}} d(\boldsymbol{u}^{\boldsymbol{k}}).$$

The conditions imposed on f (see (4)) guarantee that the integral normally converges whenever

$$|z_j|^{k_j} < \frac{1}{M} \cos\left(k_j(t_j - \arg(z_j))\right), \quad j = 1, 2,$$

and it defines a holomorphic function in the set so determined; this set contains a region $D_{\boldsymbol{t}}$ "around $\boldsymbol{t}$", namely the one given by

$$\arg(z_j) \in (t_j - \frac{\pi}{2k_j}, t_j + \frac{\pi}{2k_j}), \quad |z_j|^{k_j} < \frac{1}{M} \cos\left(k_j(t_j - \arg(z_j))\right), \quad j = 1, 2.$$

In the one variable case it is straightforward to prove, using Cauchy's theorem, that the elements of the family of holomorphic functions

$$\{(\mathcal{L}_{k,t} f, D_t)\}_{t\in(d-\theta/2, d+\theta/2)}$$

glue together to define a holomorphic function in the "sectorial region", denoted by $G(d, \theta + \frac{\pi}{k})$, that results of the union of the sets $D_{\boldsymbol{t}}$. This region is bisected by direction d and its width is $\theta + \frac{\pi}{k}$, in the sense that for every $\varepsilon > 0$ it contains a sector $S_\varepsilon = S(d, \theta + \frac{\pi}{k} - \varepsilon, \rho)$, for suitable $\rho = \rho(\varepsilon) > 0$. In our case we can apply Fubini's theorem to deduce that the functions in the family

$\{(\mathcal{L}_{\boldsymbol{k},\boldsymbol{t}}, D_{\boldsymbol{t}})\}_{\boldsymbol{t}\in(\boldsymbol{d}-\boldsymbol{\theta}/2,\boldsymbol{d}+\boldsymbol{\theta}/2)}$ again give rise to a unique holomorphic function, which we call the *Laplace transform of* f *with index* $\boldsymbol{k}$. It is defined in a polysectorial region $G(\boldsymbol{d}, \boldsymbol{\theta} + \frac{\boldsymbol{\pi}}{\boldsymbol{k}})$ which, for every $\varepsilon > 0$, contains a polysector $S_\varepsilon = S(\boldsymbol{d}, \boldsymbol{\theta} + \frac{\boldsymbol{\pi}}{\boldsymbol{k}} - \boldsymbol{\varepsilon}, \boldsymbol{\rho}_\varepsilon)$, where $\boldsymbol{\pi} = \pi \cdot \mathbf{1}$, $\boldsymbol{\varepsilon} = \varepsilon \cdot \mathbf{1}$, $\frac{\boldsymbol{\pi}}{\boldsymbol{k}} = (\pi/k_1, \pi/k_2)$ and $\boldsymbol{\rho}_\varepsilon = (\rho_1(\varepsilon), \rho_2(\varepsilon)) \in (0,\infty)^2$.

We note that the notion of strongly asymptotic expansion may be extended in an obvious way to functions defined in a polysectorial region G, by imposing the same boundedness conditions on the proper bounded subpolysectors H of polysectors S_ε contained in G.

An interesting particular case consists of the functions $f_{\boldsymbol{\theta}}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} = (\lambda_1, \lambda_2)$ and $\mathrm{Re}(\lambda_j) > 0$, $j = 1, 2$. $f_{\boldsymbol{\theta}}$ is of exponential growth at most $\boldsymbol{k}$ for every $\boldsymbol{k} \in (0,\infty)^2$, and continuous at the origin, so that $\mathcal{L}_{\boldsymbol{k}} f_{\boldsymbol{\theta}}$ makes sense; the integral expression of the Gamma function and Fubini's theorem lead to

$$(5) \qquad \mathcal{L}_{\boldsymbol{k}} f_{\boldsymbol{\theta}}(\boldsymbol{z}) = \prod_{j=1}^{2} \Gamma(1 + \lambda_j/k_j) z_j^{\lambda_j} = \Gamma(1 + \boldsymbol{\theta}/\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{\theta}}.$$

In view of this relation, we define the *formal Laplace transform with index* $\boldsymbol{k}$, denoted by $\widehat{\mathcal{L}}_{\boldsymbol{k}}$, of a formal power series $\widehat{f} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ as

$$\widehat{\mathcal{L}}_{\boldsymbol{k}} \widehat{f} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^2} a_{\boldsymbol{\alpha}} \Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{\alpha}}.$$

The next result establishes, under suitable conditions, the relation between the asymptotic behaviour of a function F in some $\mathcal{A}_{\boldsymbol{k}}(S)$ and that of its Laplace transform with index $\boldsymbol{k}$. So, we generalize to several variables the result [1, Chapter 2, Theorem 1]. Though the conditions imposed on F may seem to be quite awkward, they will turn up later on in a natural way.

We note that every $F \in \mathcal{A}_{\boldsymbol{k}}(S)$ is bounded at the origin, and so are the elements in $\mathrm{TA}(F)$.

Theorem 4.1. *Let* $\boldsymbol{k} = (k_1, k_2), \widetilde{\boldsymbol{k}} = (\widetilde{k}_1, \widetilde{k}_2) \in (0,\infty)^2$, $S = S(\boldsymbol{d}, \boldsymbol{\theta})$ *be a polysector, and* $F\colon S \to \mathbb{C}$ *be a holomorphic function such that:*
(i) $F \in \mathcal{A}_{\widetilde{\boldsymbol{k}}}(S)$, *with* $\mathrm{TA}(F) = \{h_m,\ g_n,\ a_{nm} : n, m \in \mathbb{N}\}$;
(ii) *The family* $\{\boldsymbol{u}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ *is uniformly of exponential growth at most* $\boldsymbol{k}$ *in* S, *in such a way that for every* $T \prec S$ *there exist* $r > 0$, $M > 0$, $C > 0$ *and* $B = (B_1, B_2) \in (0,\infty)^2$, *all depending on* T, *such that for* $\boldsymbol{u} \in T$ *with* $\|\boldsymbol{u}\|_\infty \geq r$ *and* $\boldsymbol{\alpha} \in \mathbb{N}^2$ *we have*

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq C\Gamma(1 + \boldsymbol{\alpha}/\widetilde{\boldsymbol{k}}) B^{\boldsymbol{\alpha}} \exp\big(M(|u_1|^{k_1} + |u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}.$$

Then, the following hold true:
(a) *The family* $\{h_m\}_{m\in\mathbb{N}}$ (*resp.* $\{g_n\}_{n\in\mathbb{N}}$) *is uniformly of exponential growth at most* k_1 *in* S_1 (*resp.* k_2 *in* S_2), *so that one may consider for every* $n, m \in \mathbb{N}$ *the functions* $\mathcal{L}_{k_1} h_m$, $\mathcal{L}_{k_2} g_n$ *and* $\mathcal{L}_{\boldsymbol{k}} \mathrm{App}_{(n,m)}(F)$.
(b) *The function* $f := \mathcal{L}_{\boldsymbol{k}} F$, *defined and analytic in a polysectorial region* $G = G(\boldsymbol{d}, \boldsymbol{\theta} + \frac{\boldsymbol{\pi}}{\boldsymbol{k}})$, *belongs to* $\mathcal{A}_{\bar{\boldsymbol{k}}}(G)$, *where* $\bar{\boldsymbol{k}} = (\bar{k}_1, \bar{k}_2)$ *is given by*

$$\frac{1}{\bar{k}_j} = \frac{1}{\widetilde{k}_j} + \frac{1}{k_j}, \quad j = 1, 2,$$

and

$$\mathrm{TA}(f) = \{\Gamma(1 + m/k_2)\mathcal{L}_{k_1} h_m,\ \Gamma(1 + n/k_1)\mathcal{L}_{k_2} g_n,\ \Gamma(1 + n/k_1)\Gamma(1 + m/k_2) a_{nm}\},$$

so that $\mathrm{App}_{\boldsymbol{\alpha}}(f) = \mathcal{L}_{\boldsymbol{k}} \mathrm{App}_{\boldsymbol{\alpha}}(F)$ *for every* $\boldsymbol{\alpha} \in \mathbb{N}^2$.

Remark 4.2. By using the same ideas as in the proof that $\mathcal{A}_{\boldsymbol{k}}(S)=\mathcal{W}_{\boldsymbol{k}}(S)$ (see [10]), the conditions in (ii) can be proved to be equivalent to the following ones: The family $\{D^{\boldsymbol{\alpha}}F(\boldsymbol{u})\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in S, in such a way that for every $T\prec S$ there exist $r>0$, $M>0$, $C>0$ and $B=(B_1,B_2)\in(0,\infty)^2$, all depending on T (and possibly distinct from those above), such that for $\boldsymbol{u}\in T$ with $\|\boldsymbol{u}\|_\infty\geq r$ we have

$$\frac{1}{\boldsymbol{\alpha}!}|D^{\boldsymbol{\alpha}}F(\boldsymbol{u})|\leq C\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big),\qquad \boldsymbol{\alpha}\in\mathbb{N}^2.$$

Remark 4.3. It is plain to see that conditions (i) and (ii) together are equivalent to the following fact: For every $T\prec S$ there exist $M>0$, $C>0$ and $B=(B_1,B_2)\in(0,\infty)^2$, all depending on T, such that for every $\boldsymbol{u}\in T$ and $\boldsymbol{\alpha}\in\mathbb{N}^2$ we have

$$|F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})|\leq C\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}.$$

Applying Remark 4.2, this amounts to the existence, for every $T\prec S$, of $M>0$, $C>0$ and $B=(B_1,B_2)\in(0,\infty)^2$ such that for every $\boldsymbol{u}\in T$ and $\boldsymbol{\alpha}\in\mathbb{N}^2$ we have

$$\frac{1}{\boldsymbol{\alpha}!}|D^{\boldsymbol{\alpha}}F(\boldsymbol{u})|\leq C\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big),\qquad \boldsymbol{\alpha}\in\mathbb{N}^2.$$

Proof. Part (a) follows directly from the previous remark and the relations in (2). Taking $\boldsymbol{\alpha}=(0,0)$ in (ii) makes clear that the function $f:=\mathcal{L}_{\boldsymbol{k}}F$ is well-defined and analytic in a polysectorial region $G=G(\boldsymbol{d},\boldsymbol{\theta}+\dfrac{\boldsymbol{\pi}}{\boldsymbol{k}})=G_1\times G_2$. It is worth noticing that the domains of definition of the families $\{\mathcal{L}_{k_1}h_m\}_m$, $\{\mathcal{L}_{k_2}g_n\}_n$ and $\{\mathcal{L}_{\boldsymbol{k}}\,\mathrm{App}_{(n,m)}(F)\}_{n,m}$ are, respectively, G_1, G_2 and G. Put

$$\mathcal{F}=\{\Gamma(1+m/k_2)\mathcal{L}_{k_1}h_m,\ \Gamma(1+n/k_1)\mathcal{L}_{k_2}g_n,\ \Gamma(1+n/k_1)\Gamma(1+m/k_2)a_{nm}\}.$$

Linearity and Fubini's theorem, together with (5), show that

$$\mathcal{L}_{\boldsymbol{k}}(\mathrm{App}_{\boldsymbol{\alpha}}(F))(\boldsymbol{z})=\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z}),\qquad \boldsymbol{\alpha}\in\mathbb{N}^2,\ \boldsymbol{z}\in G.$$

Hence we have $f(\boldsymbol{z})-\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})=\mathcal{L}_{\boldsymbol{k}}(F-\mathrm{App}_{\boldsymbol{\alpha}}(F))(\boldsymbol{z})$ for every $\boldsymbol{\alpha}\in\mathbb{N}^2$.

In what follows we will obtain suitable bounds. By (ii), given $\boldsymbol{t}=(t_1,t_2)\in(\boldsymbol{d}-\boldsymbol{\theta}/2,\boldsymbol{d}+\boldsymbol{\theta}/2)$ there exist $r>0$, $M>0$, $C>0$ and $B=(B_1,B_2)\in(0,\infty)^2$, depending on $\boldsymbol{t}$, such that for $\boldsymbol{u}=(u_1,u_2)\in S$ with $\arg(u_j)=t_j$, $j=1,2$, and such that $\|\boldsymbol{u}\|_\infty\geq r$ we have, for every $\boldsymbol{\alpha}\in\mathbb{N}^2$,

$$|F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})|\leq C\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}.$$

Since every $H\ll G$ can be covered by a finite number of sets $D_{\boldsymbol{t}}$ with $\boldsymbol{t}\in(\boldsymbol{d}-\boldsymbol{\theta}/2,\boldsymbol{d}+\boldsymbol{\theta}/2)$ (see the beginning of this section), it suffices to discuss the case $H\ll D_{\boldsymbol{t}}$, where $D_{\boldsymbol{t}}$ is given by

$$\arg(z_j)\in(t_j-\frac{\pi}{2k_j},t_j+\frac{\pi}{2k_j}),\quad |z_j|^{k_j}<\frac{1}{M}\cos\big(k_j(t_j-\arg(z_j))\big),\quad j=1,2.$$

There exist $\varepsilon_j>0$ and $\rho_j>0$, $j=1,2$, such that

$$\cos\big(k_j(t_j-\arg(z_j))\big)\geq\varepsilon_j,\quad |z_j|^{k_j}\leq\rho_j^{k_j}<\frac{\varepsilon_j}{M},\qquad j=1,2,$$

for every $\boldsymbol{z}=(z_1,z_2)\in H$, so that

$$M|z_j|^{k_j}-\cos\big(k_j(t_j-\arg(z_j))\big)\leq M\rho_j^{k_j}-\varepsilon_j=-\delta_j<0,\quad j=1,2. \tag{6}$$

From (i) we deduce there exist $D>0$ and $A=(A_1,A_2)\in(0,\infty)^2$ such that for $\boldsymbol{u}\in S$ with $\arg(u_j)=t_j$ and $|u_j|<r$, $j=1,2$, we have

$$|F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})|\leq D\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})A^{\boldsymbol{\alpha}}|\boldsymbol{u}|^{\boldsymbol{\alpha}},\qquad \boldsymbol{\alpha}\in\mathbb{N}^2.$$

Therefore, we are in conditions to write for every $\boldsymbol{z} \in H$

$$\begin{aligned}
|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})| &= |\mathcal{L}_{\boldsymbol{k}}(F - \mathrm{App}_{\boldsymbol{\alpha}}(F))(\boldsymbol{z})| \\
&= \left| \frac{1}{\boldsymbol{z}^{\boldsymbol{k}}} \int_{(0,\infty)^2(\boldsymbol{t})} (F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})) \exp(-(\boldsymbol{u}/\boldsymbol{z})^{\boldsymbol{k}}) d(\boldsymbol{u}^{\boldsymbol{k}}) \right| \\
&\leq |z_1|^{-k_1} |z_2|^{-k_2} \int_{(0,\infty)^2} |F(s_1 e^{it_1}, s_2 e^{it_2}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(s_1 e^{it_1}, s_2 e^{it_2})| \times \\
&\times \exp\Big(- \sum_{j=1}^{2} \frac{s_j^{k_j}}{|z_j|^{k_j}} \cos(k_j(t_j - \arg(z_j)))\Big) \prod_{j=1}^{2} k_j s_j^{k_j - 1} ds_1\, ds_2 \\
&\leq k_1 k_2 |\boldsymbol{z}|^{-\boldsymbol{k}} \Bigg[\int_{(0,r)^2} D\Gamma(1 + \boldsymbol{\alpha}/\widetilde{\boldsymbol{k}}) A^{\boldsymbol{\alpha}} \boldsymbol{s}^{\boldsymbol{\alpha}+\boldsymbol{k}-\mathbf{1}} \exp\Big(- \sum_{j=1}^{2} \varepsilon_j s_j^{k_j} / |z_j|^{k_j} \Big) d\boldsymbol{s} \\
&+ \int_{(0,\infty)^2 \setminus (0,r)^2} C\Gamma(1 + \boldsymbol{\alpha}/\widetilde{\boldsymbol{k}}) B^{\boldsymbol{\alpha}} \boldsymbol{s}^{\boldsymbol{\alpha}+\boldsymbol{k}-\mathbf{1}} \exp\Big(\sum_{j=1}^{2} M s_j^{k_j} - \varepsilon_j s_j^{k_j} |z_j|^{k_j} \Big) d\boldsymbol{s} \Bigg].
\end{aligned}$$

One may extend both integrals to $\mathbb{R}^2$, take account of (6) and of Stirling's formula in order to arrive at

$$\begin{aligned}
|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})| \leq \Big(DA^{\boldsymbol{\alpha}} \frac{\varepsilon_1^{-n/k_1} \varepsilon_2^{-m/k_2}}{\varepsilon_1 \varepsilon_2} + CB^{\boldsymbol{\alpha}} \frac{\delta_1^{-n/k_1} \delta_2^{-m/k_2}}{\delta_1 \delta_2} \Big) \Gamma(1 + \boldsymbol{\alpha}/\widetilde{\boldsymbol{k}}) \Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) |\boldsymbol{z}|^{\boldsymbol{\alpha}} \\
\leq C_0 A_0^{\boldsymbol{\alpha}} \Gamma(1 + \boldsymbol{\alpha}/\bar{\boldsymbol{k}}) |\boldsymbol{z}|^{\boldsymbol{\alpha}},
\end{aligned}$$

for suitably large $C_0 > 0$ and $A_0 \in (0, \infty)^2$. □

Remark 4.4. There are some special cases to be discussed. For $\widetilde{k}_2 \in (0, \infty)$ and a polysector $S = S(\boldsymbol{d}, \boldsymbol{\theta})$ let us define the space $\mathcal{A}_{(\infty, \widetilde{k}_2)}(S)$ consisting of the functions $F \in \mathcal{A}(S)$ such that for every $T \ll S$ there exist $C_T > 0$ and $A_T \in (0, \infty)^2$ so that

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq C_T \Gamma(1 + m/\widetilde{k}_2) A_T^{\boldsymbol{\alpha}} |\boldsymbol{u}|^{\boldsymbol{\alpha}}, \qquad \boldsymbol{u} \in T,\ \boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2.$$

Take $F \in \mathcal{A}_{(\infty, \widetilde{k}_2)}(S)$ and put $\mathrm{TA}(F) = \{h_m,\, g_n,\, a_{nm}\}$. One can easily show that for every $m \in \mathbb{N}$ the series $\widehat{h}_m = \sum_{n=0}^{\infty} a_{nm} u_1^n$, asymptotic expansion of h_m, converges to h_m in some disk around 0, independent from m. Also, for every $n \in \mathbb{N}$ we have $g_n \in \mathcal{A}_{\widetilde{k}_2}(S_2)$.

Now, suppose F also satisfies that for every $T \prec S$ there exist $r > 0$, $M > 0$, $C > 0$ and $B = (B_1, B_2) \in (0, \infty)^2$, all depending on T, such that for $\boldsymbol{u} \in T$ with $\|\boldsymbol{u}\|_\infty \geq r$ and for every $\boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2$ we have

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq C\Gamma(1 + m/\widetilde{k}_2) B^{\boldsymbol{\alpha}} \exp\big(M(|u_1|^{k_1} + |u_2|^{k_2})\big) |\boldsymbol{u}|^{\boldsymbol{\alpha}}.$$

The proof of the previous result remains valid and we conclude that $f = \mathcal{L}_{\boldsymbol{k}}(F) \in \mathcal{A}_{\bar{\boldsymbol{k}}}(G)$, where $\bar{\boldsymbol{k}} = (k_1, (\widetilde{k}_2^{-1} + k_2^{-1})^{-1})$.

Remark 4.5. In a similar way, we may consider $\mathcal{A}_{(\infty,\infty)}(S)$, the subset of $\mathcal{A}(S)$ of those F such that for every $T \ll S$ there exist $C_T > 0$ and $A_T \in (0, \infty)^2$ so that

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq C_T A_T^{\boldsymbol{\alpha}} |\boldsymbol{u}|^{\boldsymbol{\alpha}}, \qquad \boldsymbol{u} \in T,\ \boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2.$$

For $F \in \mathcal{A}_{(\infty,\infty)}(S)$ with $\mathrm{TA}(F) = \{h_m,\, g_n,\, a_{nm}\}$, one has that for every $m, n \in \mathbb{N}$ the series $\widehat{h}_m = \sum_{n=0}^{\infty} a_{nm} u_1^n$ (resp. $\widehat{g}_n = \sum_{m=0}^{\infty} a_{nm} u_2^m$), asymptotic expansion of h_m (resp. g_n), converges to h_m

(resp. g_n) in some disk around 0, independent from m (resp. n). As for FA(F), it converges to F in some polydisk. Moreover, suppose that for every $T \prec S$ there exist $r > 0$, $M > 0$, $C > 0$ and $B = (B_1, B_2) \in (0,\infty)^2$ such that for $\boldsymbol{u} \in T$ with $\|\boldsymbol{u}\|_\infty \geq r$,

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq CB^{\boldsymbol{\alpha}} \exp\big(M(|u_1|^{k_1} + |u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}, \ \ \boldsymbol{\alpha} = (n,m) \in \mathbb{N}^2.$$

Then, we obtain that $f = \mathcal{L}_{\boldsymbol{k}}(F) \in \mathcal{A}_{\boldsymbol{k}}(G)$.

Remark 4.6. If we simply assume that $F \in \mathcal{A}(S)$ and that the family $\{\boldsymbol{z}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{z})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in S, we would obtain that $f = \mathcal{L}_{\boldsymbol{k}}F \in \mathcal{A}(G)$, with TA$(f)$ as indicated above.

It is interesting to note that for one-variable functions the previous theorem needs hypothesis (i) and f being of exponential growth at most k, while the rest of hyphotesis (ii) is deduced in the course of the proof. We have not been able to do the same in this situation. Anyway, the inclusion of hyphoteses in (ii) will be justified by the forthcoming result on Borel transforms, Theorem 4.8.

The following statement stems from its analogue in one variable and Fubini's theorem.

Theorem 4.7. *Let $F\colon S \to \mathbb{C}$ be a holomorphic function, bounded at the origin and of exponential growth at most $\boldsymbol{k}$ in S. Suppose $\mathcal{L}_{\boldsymbol{k}}F$ identically vanishes. Then so does F.*

We now turn our attention to the generalization of the Borel transform for functions of two variables.

First, assume that (piecewise continuously differentiable) paths $\gamma_j\colon I_j \to \mathcal{R}$ are given for $j = 1, 2$ (I_j is an interval in the real line). Then, we make explicit the meaning we give to the integral of a continuous function f, defined in $\boldsymbol{\gamma}^* = \prod_{j=1}^2 \gamma_j^* \subset \mathcal{R}^2$, over the "product path" $\boldsymbol{\gamma} = \prod_{j=1}^2 \gamma_j$:

$$\int_{\boldsymbol{\gamma}} f(\boldsymbol{z})\, d\boldsymbol{z} := \int_{\prod_{j=1}^2 I_j} f(\gamma_1(s_1), \gamma_2(s_2))\gamma_1'(s_1)\gamma_2'(s_2)\, ds_1\, ds_2.$$

We note that the value so defined does not depend on the concrete (equivalent, up to diffeomorphism) parameterization given for the geometric curves represented by the γ_j.

Secondly, for $k > 0$ and $t \in \mathbb{R}$ we introduce the paths $\gamma_k(t)$ in $\mathcal{R}$, that consist of a segment from the origin to a point z_0 with $\arg(z_0) = t + \pi/(2k) + \varepsilon/(2k)$ (for some $\varepsilon \in (0,\pi)$), then the circular arc $|z| = |z_0|$ from z_0 to the point z_1 on the ray $\arg(z) = t - \pi/(2k) - \varepsilon/(2k)$, and finally the segment from z_1 to the origin. We note that for a sector $S = S(d,\theta,\rho)$ with $\theta > \pi/k$ and for a ray $\arg(z) = t$ in S such that $|t - d| < (\theta - \pi/k)/2$, one can always choose a path $\gamma_k(t)$ contained in S.

Suppose $S = S(\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\rho})$ is a polysector in $\mathcal{R}^2$, and $\boldsymbol{k} \in (0,\infty)^2$ is such that $\theta_j > \pi/k_j$ for $j = 1, 2$ (for short, $\boldsymbol{\theta} > \boldsymbol{\pi}/\boldsymbol{k}$). Take $\boldsymbol{t} \in \mathbb{R}^2$ such that $|t_j - d_j| < (\theta_j - \pi/k_j)/2$, and choose a path $\gamma_{k_j}(t_j)$ (as above) contained in S_j; the product path $\prod_{j=1}^2 \gamma_{k_j}(t_j)$ will be denoted by $\boldsymbol{\gamma}_{\boldsymbol{k}}(\boldsymbol{t})$. Given a function $f\colon S \to \mathbb{C}$, analytic and bounded at the origin, we define the *Borel transform of f with index $\boldsymbol{k}$ in direction $\boldsymbol{t}$* as

$$\mathcal{B}_{\boldsymbol{k},\boldsymbol{t}}f(\boldsymbol{u}) = \frac{1}{(2\pi i)^2}\int_{\boldsymbol{\gamma}_{\boldsymbol{k}}(\boldsymbol{t})} \boldsymbol{z}^{\boldsymbol{k}} f(\boldsymbol{z})\, \exp\big((\boldsymbol{u}/\boldsymbol{z})^{\boldsymbol{k}}\big)\, d(\boldsymbol{z}^{-\boldsymbol{k}}),$$

where the integrand must be understood to be

$$\boldsymbol{z}^{\boldsymbol{k}} f(\boldsymbol{z})\, \exp\big(\textstyle\sum_{j=1}^2 (u_j/z_j)^{k_j}\big)\, \prod_{j=1}^2 (-k_j z_j^{-k_j-1} dz_j).$$

The integral is seen to converge normally in the polysector $S(\boldsymbol{t},\boldsymbol{\varepsilon}/\boldsymbol{k})$, so that $\mathcal{B}_{\boldsymbol{k},\boldsymbol{t}}f$ is holomorphic there. In the one variable case, and by Cauchy's theorem, we first deduce that the value of this function does not depend on either ε or z_0, and second that the family

$$\{\big(\mathcal{B}_{k,t}f, S(t,\varepsilon/k)\big)\}_{\{t\colon |t-d|<(\theta-\pi/k)/2\}}$$

defines a unique function $\mathcal{B}_k f\colon S(d,\theta-\pi/k)\to\mathbb{C}$. In the case we are dealing with, we may apply Fubini's theorem to obtain that the family $\{\big(\mathcal{B}_{\boldsymbol{k},\boldsymbol{t}}f, S(\boldsymbol{t},\boldsymbol{\varepsilon})\big)\}_{\boldsymbol{t}}$ determines a function $\mathcal{B}_{\boldsymbol{k}}f$, defined and holomorphic in $S(\boldsymbol{d},\boldsymbol{\theta}-\boldsymbol{\pi}/\boldsymbol{k})$, which we call the *Borel transform of f with index $\boldsymbol{k}$*.

The functions $f_{\boldsymbol{\theta}}(\boldsymbol{z})=\boldsymbol{z}^{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}=(\lambda_1,\lambda_2)$ and $\mathrm{Re}(\lambda_j)>0$, $j=1,2$, are bounded at the origin; Fubini's theorem and Hankel's integral for the function $1/\Gamma$ imply that

$$\mathcal{B}_{\boldsymbol{k}}f_{\boldsymbol{\theta}}(\boldsymbol{u})=\prod_{j=1}^{2}u_j^{\lambda_j}/\Gamma(1+\lambda_j/k_j)=\boldsymbol{u}^{\boldsymbol{\theta}}/\Gamma(1+\boldsymbol{\theta}/\boldsymbol{k}).$$

So, the *formal Borel transform with index $\boldsymbol{k}$*, denoted by $\widehat{\mathcal{B}}_{\boldsymbol{k}}$, of a formal power series $\widehat{f}=\sum_{\boldsymbol{\alpha}\in\mathbb{N}^2}a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$ is defined as

$$\widehat{\mathcal{B}}_{\boldsymbol{k}}\widehat{f}=\sum_{\boldsymbol{\alpha}\in\mathbb{N}^2}a_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}/\Gamma(1+\boldsymbol{\alpha}/\boldsymbol{k}).$$

It is obvious that $\widehat{f}\in\mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ if and only if $\widehat{\mathcal{B}}_{\boldsymbol{k}}\widehat{f}\in\mathbb{C}\{\boldsymbol{z}\}$.

The following result is along the same lines as Theorem 4.1, now for the Borel transform. We note that every $f\in\mathcal{A}_{\boldsymbol{k}}(S)$, as well as the elements in $\mathrm{TA}(f)$, are bounded at the origin.

Theorem 4.8. *Let $G=G(\boldsymbol{d},\boldsymbol{\theta})$ be a polysectorial region, $\boldsymbol{k}=(k_1,k_2)\in(0,\infty)^2$ be such that $\boldsymbol{\theta}>\boldsymbol{\pi}/\boldsymbol{k}$, and $f\colon G\to\mathbb{C}$ be a holomorphic function belonging to $\mathcal{A}_{\bar{\boldsymbol{k}}}(G)$, with $\bar{\boldsymbol{k}}=(\bar{k}_1,\bar{k}_2)\in(0,\infty)^2$ and $\mathrm{TA}(f)=\{h_m,g_n,a_{nm}\}$.*
Let us define $\widetilde{\boldsymbol{k}}=(\widetilde{k}_1,\widetilde{k}_2)$ as

$$\widetilde{k}_j=\begin{cases}\Big(\dfrac{1}{\bar{k}_j}-\dfrac{1}{k_j}\Big)^{-1} & \textit{if } \bar{k}_j<k_j,\\ \infty & \textit{if } \bar{k}_j\geq k_j,\end{cases}\qquad j=1,2.$$

Then, we have:
(a) *$F:=\mathcal{B}_{\boldsymbol{k}}f\in\mathcal{A}_{\widetilde{\boldsymbol{k}}}(\widetilde{S})$, where $\widetilde{S}=S(\boldsymbol{d},\boldsymbol{\theta}-\boldsymbol{\pi}/\boldsymbol{k})$, and*

$$\mathrm{TA}(F)=\{\frac{\mathcal{B}_{k_1}h_m}{\Gamma(1+m/k_2)},\frac{\mathcal{B}_{k_2}g_n}{\Gamma(1+n/k_1)},\frac{a_{nm}}{\Gamma(1+n/k_1)\Gamma(1+m/k_2)}\},$$

so that $\mathrm{App}_{\boldsymbol{\alpha}}(F)=\mathcal{B}_{\boldsymbol{k}}\,\mathrm{App}_{\boldsymbol{\alpha}}(f)$ for every $\boldsymbol{\alpha}\in\mathbb{N}^2$.
(b) *The family $\{\boldsymbol{u}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in $\widetilde{S}$, in such a way that for every $\widetilde{T}\prec\widetilde{S}$ there exist $r>0$, $M>0$, $D>0$ and $B=(B_1,B_2)\in(0,\infty)^2$, all depending on $\widetilde{T}$, such that for $\boldsymbol{u}\in\widetilde{T}$ with $\|\boldsymbol{u}\|_\infty\geq r$ and for $\boldsymbol{\alpha}\in\mathbb{N}^2$ we have*

$$|F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})|\leq D\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}.$$

Proof. According to Remark 4.3, it is enough to prove that for every $\widetilde{T}\prec\widetilde{S}$ there exist $M>0$, $D>0$ and $B=(B_1,B_2)\in(0,\infty)^2$ such that for every $\boldsymbol{u}\in\widetilde{T}$ and $\boldsymbol{\alpha}\in\mathbb{N}^2$ we have

$$|F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})|\leq D\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}}\exp\big(M(|u_1|^{k_1}+|u_2|^{k_2})\big)|\boldsymbol{u}|^{\boldsymbol{\alpha}}.\tag{7}$$

Observe that $\mathcal{B}_{k_1}h_m$ and $\mathcal{B}_{k_2}g_n$ are well defined for every $m,n\in\mathbb{N}$; let us put

$$\mathcal{F}=\{\frac{\mathcal{B}_{k_1}h_m}{\Gamma(1+m/k_2)},\frac{\mathcal{B}_{k_2}g_n}{\Gamma(1+n/k_1)},\frac{a_{nm}}{\Gamma(1+n/k_1)\Gamma(1+m/k_2)}\}.$$

Fubini's theorem implies that $\mathcal{B}_{\boldsymbol{k}}\,\mathrm{App}_{\boldsymbol{\alpha}}(f)=\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})$, so that $F(\boldsymbol{u})-\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{u})=\mathcal{B}_{\boldsymbol{k}}(f-\mathrm{App}_{\boldsymbol{\alpha}}(f))(\boldsymbol{u})$, for every $\boldsymbol{\alpha}\in\mathbb{N}^2$ and every $\boldsymbol{u}\in\widetilde{S}$.

Since every unbounded proper subpolysector of $\widetilde{S}$ may be covered by a finite number of polysectors of the form $\widetilde{T}=\widetilde{T}_1(\tau_1,\varepsilon/(2k_1))\times\widetilde{T}_2(\tau_2,\varepsilon/(2k_2))$, with $|\tau_j-d_j|<(\theta_j-\pi/k_j)/2$, $j=1,2$, we will only consider this special $\widetilde{T}$. Choose a product path $\boldsymbol{\gamma_k}(\boldsymbol{\tau})=\prod_{j=1}^2\gamma_{k_j}(\tau_j)$ (see the definition of $\mathcal{B}_{\boldsymbol{k}}$), whose support will be contained in a certain

$$T=T_1(\tau_1,(\varepsilon+\pi)/(2k_1),\rho_1)\times T_2(\tau_2,(\varepsilon+\pi)/(2k_2),\rho_2)\ll G,$$

so that for every $\boldsymbol{u}\in\widetilde{T}$ we can write $\mathcal{B}_{\boldsymbol{k}}(f-\mathrm{App}_{\boldsymbol{\alpha}}(f))(\boldsymbol{u})$ as

$$\frac{1}{(2\pi i)^2}\int_{\boldsymbol{\gamma_k}(\boldsymbol{\tau})}\boldsymbol{z}^{\boldsymbol{k}}(f(\boldsymbol{z})-\mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z}))\exp\left((\boldsymbol{u}/\boldsymbol{z})^{\boldsymbol{k}}\right)d(\boldsymbol{z}^{-\boldsymbol{k}}).$$

On one hand, there exist $C=C(T)>0$ and $A=A(T)=(A_1,A_2)\in(0,\infty)^2$ such that

$$|f(\boldsymbol{z})-\mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})|\le C\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}|\boldsymbol{z}|^{\boldsymbol{\alpha}},\quad \boldsymbol{z}\in T,\ \boldsymbol{\alpha}\in\mathbb{N}^2.$$

On the other hand, we have $\gamma_{k_j}(\tau_j)=\sum_{\ell=1}^3\gamma_{k_j}^{\ell}(\tau_j)$, where: $\gamma_{k_j}^2(\tau_j)$ is the circular arc, centered at 0 and with (suitable, to be chosen) radius $r_j\in(0,\rho_j)$, that goes from the ray $\arg(z_j)=\tau_j+(\varepsilon+\pi)/(2k_j)$ to the ray $\arg(z_j)=\tau_j-(\varepsilon+\pi)/(2k_j)$, for some $0<\varepsilon<\pi$; and $\gamma_{k_j}^1(\tau_j)$ (resp. $\gamma_{k_j}^3(\tau_j)$) is the segment from 0 to the initial point of $\gamma_{k_j}^2(\tau_j)$ (resp. from the end point of $\gamma_{k_j}^2(\tau_j)$ to the origin). Formally, it may be said that

$$\boldsymbol{\gamma_k}(\boldsymbol{\tau})=\prod_{j=1}^2\gamma_{k_j}(\tau_j)=\prod_{j=1}^2\sum_{\ell=1}^3\gamma_{k_j}^{\ell}(\tau_j)=\sum_{\sigma\in 3^{\{1,2\}}}\prod_{j=1}^2\gamma_{k_j}^{\sigma(j)}(\tau_j);$$

after parameterization, it may be justified that the original integral is the sum of the nine corresponding integrals I_σ. In order to obtain the desired estimates, different cases will be studied.

For $\sigma\equiv\{\sigma(1),\sigma(2)\}$ equal to $\{1,1\}$, $\{1,3\}$, $\{3,1\}$ or $\{3,3\}$, one easily shows that

$$\begin{aligned}|I_\sigma| &\le k_1k_2C\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}\int_0^{r_1}t_1^{n-1}\exp\Big(-c(|u_1|/t_1)^{k_1}\Big)dt_1\\ &\times\int_0^{r_2}t_2^{m-1}\exp\Big(-c(|u_2|/t_2)^{k_2}\Big)dt_2\\ &\le\frac{C}{c^2}\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}r_1^nr_2^m(r_1/|u_1|)^{k_1}(r_2/|u_2|)^{k_2}\\ &\times\exp\Big(-c((|u_1|/r_1)^{k_1}+(|u_2|/r_2)^{k_2})\Big),\end{aligned}$$

where $c=\sin(\varepsilon/4)>0$.

For σ equal to $\{1,2\}$ or $\{3,2\}$, one has

$$\begin{aligned}|I_\sigma| &\le k_1k_2C\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}\int_0^{r_1}t_1^{n-1}\exp\Big(-c(|u_1|/t_1)^{k_1}\Big)dt_1\\ &\times\int_{\tau_2-(\varepsilon+\pi)/(2k_2)}^{\tau_2+(\varepsilon+\pi)/(2k_2)}r_2^m\exp\Big((|u_2|/r_2)^{k_2}\Big)dt_2\\ &\le\frac{2\pi C}{c}\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}r_1^nr_2^m(r_1/|u_1|)^{k_1}\\ &\times\exp\Big(-c(|u_1|/r_1)^{k_1}\Big)\exp\Big((|u_2|/r_2)^{k_2}\Big).\end{aligned}$$

Similarly, when σ equals $\{2,1\}$ or $\{2,3\}$,

$$|I_\sigma|\le\frac{2\pi C}{c}\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}}r_1^nr_2^m(r_2/|u_2|)^{k_2}\exp\Big((|u_1|/r_1)^{k_1}\Big)\exp\Big(-c(|u_2|/r_2)^{k_2}\Big).$$

Finally, for $\sigma = \{2,2\}$ we get

$$|I_\sigma| \leq 4\pi^2 C\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}} r_1^n r_2^m \exp\Big((|u_1|/r_1)^{k_1} + (|u_2|/r_2)^{k_2}\Big).$$

Summing up, we get

$$(8)\quad |F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{u})| = |\mathcal{B}_{\boldsymbol{k}}(f - \mathrm{App}_{\boldsymbol{\alpha}}(f))(\boldsymbol{u})| \leq \frac{C}{c^2}\Gamma(1+\boldsymbol{\alpha}/\bar{\boldsymbol{k}})A^{\boldsymbol{\alpha}} r_1^n r_2^m \times \prod_{j=1}^{2}[(r_j/|u_j|)^{k_j}\exp(-c(|u_j|/r_j)^{k_j}) + \exp((|u_j|/r_j)^{k_j})].$$

This bound entails a separation of variables, so, after dropping the subindices, in order to obtain (7) it will be enough to prove that there exist $D, B, M > 0$ such that for every $t > 0$ and every $n \in \mathbb{N}_0$ one has

$$g(n,t) := \inf_{0<r<\rho} \Gamma(1+n/\bar{k})r^n[(r/t)^k\exp(-c(t/r)^k) + \exp((t/r)^k)] \leq DB^n\Gamma(1+n/\tilde{k})\exp(Mt^k)t^n.$$

Let us fix $r_0 \in (0,\rho)$, and let us first reason for $t \in (0, r_0]$. Choose a constant $E > 0$ such that $Er_0k^{1/k} < \rho$, and consider for $n \geq 1$ the radius $r = Et(k/n)^{1/k} \leq E\rho k^{1/k} < \rho$. One obtains that

$$g(n,t) \leq \Gamma(1+n/\bar{k})E^n t^n (k/n)^{n/k}\Big(\frac{kE^k}{n}\exp\big(-cn/(kE^k)\big) + \exp\big(n/(kE^k)\big)\Big).$$

Stirling's formula implies that for suitable $\widetilde{C}, D, \widetilde{A}, B > 0$,

$$(9)\qquad g(n,t) \leq \widetilde{C}\widetilde{A}^n\frac{\Gamma(1+n/\bar{k})}{\Gamma(1+n/k)}t^n \leq DB^n\Gamma(1+n/\widetilde{k})t^n,$$

where $\widetilde{k}$ is as defined in the statement (in case some $\widetilde{k}_j$ is assigned the value ∞, the corresponding Gamma factor is understood to be equal to 1).

In case $n = 0$, we take $E > 0$ such that $Er_0 < \rho$, and consider the radius $r = Et < \rho$, for which one deduces that

$$g(0,t) \leq E^k\exp(-c/E^k) + \exp(1/E^k).$$

Observe that we may already say that $F \in \mathcal{A}_{\widetilde{\boldsymbol{k}}}(\widetilde{S})$ and $\mathrm{TA}(F) = \mathcal{F}$ (when some $\widetilde{k}_j$ is assigned the value ∞, the space is the one introduced in the remarks following the proof of Theorem 4.1).

We proceed to the case $t > r_0$. Whenever $n \in \mathbb{N}$ is such that $r := t(k/n)^{1/k} < \rho$, we may consider this radius to write

$$g(n,t) \leq \Gamma(1+n/\bar{k})t^n(k/n)^{n/k}\Big(\frac{k}{n}\exp(-cn/k) + \exp(n/k)\Big),$$

and we conclude as in (9). Otherwise, we would have $t(k/n)^{1/k} \geq \rho$, or in other words, $r_0^n \leq \rho^n \leq t^n(k/n)^{n/k}$. Now we choose r_0 as the radius and deduce that

$$\begin{aligned} g(n,t) &\leq \Gamma(1+n/\bar{k})r_0^n\Big(\frac{r_0}{t}\exp\big(-c(t/r_0)^k\big) + \exp\big((t/r_0)^k\big)\Big) \\ &\leq 2\Gamma(1+n/\bar{k})r_0^n\exp\big((t/r_0)^k\big) \leq 2\Gamma(1+n/\bar{k})t^n(k/n)^{n/k}\exp(Mt^k). \end{aligned}$$

The conclusion is immediately reached. □

Remark 4.9. It is worth mentioning that inequality (9) supplies more specific information in case some $\bar{k}_j$ is strictly greater than k_j. If, say, $\bar{k}_1 > k_1$ and $\bar{k}_2 < k_2$, then not only $F := \mathcal{B}_{\boldsymbol{k}} f$ belongs to $\mathcal{A}_{(\infty,\widetilde{k}_2)}(\widetilde{S})$ (with the consequences this entails for $\mathrm{TA}(F) = \{\widetilde{h}_m = \frac{\mathcal{B}_{k_1} h_m}{\Gamma(1+m/k_2)}, \widetilde{g}_n = \frac{\mathcal{B}_{k_2} g_n}{\Gamma(1+n/k_1)}, \widetilde{a}_{nm} = \frac{a_{nm}}{\Gamma(1+n/k_1)\Gamma(1+m/k_2)}\}$), but we also deduce that $\widetilde{h}_m$ is in fact an entire function with Taylor series $\widetilde{h}_m(u_1) = \sum_{n=0}^{\infty} \widetilde{a}_{nm} u_1^n$, and of exponential growth at most $(k_1^{-1} - \bar{k}_1^{-1})^{-1}$.

In the case $\bar{k}_1 > k_1$ and $\bar{k}_2 > k_2$, we have that F, $\widetilde{h}_m$ and $\widetilde{g}_n$ are all entire functions, and $F = \mathrm{FA}(F) = \sum_{n,m} \widetilde{a}_{nm} u_1^n u_2^m$, $\widetilde{h}_m(u_1) = \sum_{n=0}^{\infty} \widetilde{a}_{nm} u_1^n$, $m \in \mathbb{N}$, and $\widetilde{g}_n(u_2) = \sum_{m=0}^{\infty} \widetilde{a}_{nm} u_2^m$, $n \in \mathbb{N}$. The h_m (resp. the g_n) are of exponential growth at most $(k_1^{-1} - \bar{k}_1^{-1})^{-1}$ (resp. $(k_2^{-1} - \bar{k}_2^{-1})^{-1}$). As for F, its growth is at most $((k_1^{-1} - \bar{k}_1^{-1})^{-1}, (k_2^{-1} - \bar{k}_2^{-1})^{-1})$, as can be deduced from the following result, that is based on the ideas in [12, Chapter 5] and whose proof is elementary (see also [2, Theorem 69]).

Proposition 4.10. *Let $G\colon \mathbb{C}^2 \to \mathbb{C}$ be an entire function, with*

$$G(u_1, u_2) = \sum_{n,m=0}^{\infty} b_{nm} u_1^n u_2^m.$$

Then, G is of exponential growth at most $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$ if and only if

$$\liminf_{n+m\to\infty} \frac{\log\left(\frac{1}{|b_{nm}|}\right)}{\log\left[n^{n/k_1} m^{m/k_2}\right]} \geq 1.$$

Remark 4.11. We also point out that if we simply assume that $f \in \mathcal{A}(G)$, then $F := \mathcal{B}_{\boldsymbol{k}} f \in \mathcal{A}(\widetilde{S})$, with the same relation linking the corresponding total families, and $\{\boldsymbol{u}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in $\widetilde{S}$.

We close this section indicating that Laplace and Borel transforms are the inverse of each other. The proof of the first part of the following result has been done in Theorem 4.8, while the second part is obtained along the same lines as Theorem 24 in [2].

Theorem 4.12. *Let f be holomorphic in a polysectorial region $G = G(\boldsymbol{d}, \boldsymbol{\theta})$ and bounded at the origin. If $\boldsymbol{k} > \dfrac{\boldsymbol{\pi}}{\boldsymbol{\theta}}$, then $F := \mathcal{B}_{\boldsymbol{k}} f$ is holomorphic in the polysector $S = S(\boldsymbol{d}, \boldsymbol{\theta} - \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}})$, bounded at the origin and of exponential growth at most $\boldsymbol{k}$ in S, and its Laplace transform with index $\boldsymbol{k}$, defined in a polysectorial region $\widetilde{G} = \widetilde{G}(\boldsymbol{d}, \widetilde{\boldsymbol{\theta}})$, agrees with f in $G \cap \widetilde{G}$.*

From Theorems 4.1, 4.12 and 4.7 we easily deduce

Theorem 4.13. *Let F be holomorphic in a polysector $S = S(\boldsymbol{d}, \boldsymbol{\theta})$, bounded at the origin and of exponential growth at most $\boldsymbol{k}$ in S. Then, $f := \mathcal{L}_{\boldsymbol{k}} F$ is holomorphic in a polysectorial region $G = G(\boldsymbol{d}, \boldsymbol{\theta} + \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}})$ and bounded at the origin, and its Borel transform with index $\boldsymbol{k}$ agrees with F in S.*

5 $\boldsymbol{k}$-Summability in a direction

Throughout this section we will tacitly use the results concerning k-summability in a direction of formal power series in one variable. They all may be found in [1, Chapter 3].

Definition 5.1. Let $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$ and $\boldsymbol{d} = (d_1, d_2) \in \mathbb{R}^2$. A formal power series $\widehat{f} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ is said to be ***k**-summable in direction **d*** if there exist a polysector $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$, with $\boldsymbol{\theta} > \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}}$, and a function $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ such that $\mathrm{FA}(f) = \widehat{f}$.

From Proposition 3.3 we see that such f is unique, and it is called the ***k**-sum of $\widehat{f}$ in direction **d***; we write $f = \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}(\widehat{f})$. Also, as already indicated, in this situation we have $\widehat{f} \in \mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$.

In order to understand the relation of this concept with the classical summation procedure for one variable functions, we need the theory of (Gevrey) asymptotic expansions for holomorphic functions of one variable with values in a complex Banach space E, as well as that of summability of formal power series in one variable with coefficients in E. The basic results of both theories remain unaltered in this case, as it can be seen in [2, Chapters 4–6].

Let $(E, \|\cdot\|)$ be a complex Banach space, $S = S_1 \times S_2$ a polysector in $\mathcal{R}^2$, $\boldsymbol{k} = (k_1, k_2) \in (0,\infty)^2$ and $B = (B_1, B_2) \in (0,\infty)^2$. Denote by $\mathcal{W}_{\boldsymbol{k},B}(S, E)$ the vector space consisting of the holomorphic functions $f \colon S \to E$ such that

$$\|f\|_{\boldsymbol{k},B} := \sup_{\boldsymbol{z}\in S,\ \boldsymbol{\alpha}\in\mathbb{N}^2} \frac{\|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\|}{\boldsymbol{\alpha}!\,\Gamma(1+\boldsymbol{\alpha}/\boldsymbol{k})B^{\boldsymbol{\alpha}}} < +\infty.$$

$(\mathcal{W}_{\boldsymbol{k},B}(S,E), \|\cdot\|_{\boldsymbol{k},B})$ is a Banach space. Of course, this definition may be trivially adapted for functions of any number of variables.

Proposition 5.2. *Let S, E, $\boldsymbol{k}$ and B be as before. Then, the map*

$$f \in \mathcal{W}_{\boldsymbol{k},B}(S,E) \to f^* \in \mathcal{W}_{k_1,B_1}(S_1, \mathcal{W}_{k_2,B_2}(S_2,E))$$

defined for every $z_1 \in S_1$ by $f^*(z_1) = f(z_1,\cdot)$ is an isomorphism. We have $\|f^*(z_1)\|_{k_2,B_2} \le \|f\|_{\boldsymbol{k},B}$, $\|f^*\|_{k_1,B_1} = \|f\|_{\boldsymbol{k},B}$ and, for every $n,m \in \mathbb{N}$,

$$D^{(n,m)} f(z_1,z_2) = D^m(D^n f^*(z_1))(z_2), \qquad (z_1,z_2) \in S. \tag{10}$$

It is easy to see that $\mathcal{W}_{\boldsymbol{k},B}(S,E) \subset \mathcal{W}_{\boldsymbol{k}}(S,E) = \mathcal{A}_{\boldsymbol{k}}(S,E)$, and that the restriction to any $T \ll S$ of the functions in $\mathcal{A}_{\boldsymbol{k}}(S,E)$ provides elements of $\mathcal{W}_{\boldsymbol{k},B}(T,E)$, for some adequately chosen $B = B(T)$.

We will apply the previous result (with $E = \mathbb{C}$) in the following way: given a $\boldsymbol{k}$-summable power series in direction $\boldsymbol{d}$, $\widehat{f}$, we can write

$$\widehat{f} = \sum_{n,m=0}^{\infty} a_{nm} z_1^n z_2^m == \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{nm} z_2^m \right) z_1^n = \sum_{n=0}^{\infty} \widehat{g}_n z_1^n.$$

Now, for $f = \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}(\widehat{f}) \in \mathcal{A}_{\boldsymbol{k}}(S)$, with $S = S(\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\rho}) = S_1 \times S_2$ and $\boldsymbol{\theta} > \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}}$, we have $\mathrm{TA}(f) = \{h_m,\ g_n,\ a_{nm} \colon n,m \in \mathbb{N}\}$, where $g_n \in \mathcal{A}_{k_2}(S_2)$ is such that $\mathrm{TA}(g_n) \equiv \mathrm{FA}(g_n) = \widehat{g}_n$, $n \in \mathbb{N}$. Since the width of S_2 is greater than π/k_2, this means that every $\widehat{g}_n$ is k_2-summable in direction d_2, and that the sums g_n are all defined in a common sector S_2 around d_2. Also, from (2) and the uniformity in those limits we obtain that for every $n,m \in \mathbb{N}$

$$g_n^{(m)}(z_2) = \lim_{\substack{z_1\to 0\\ z_1 \in T_1 \ll S_1}} \frac{D^{(n,m)} f(z_1,z_2)}{n!}. \tag{11}$$

Choose $T = T_1 \times T_2 \ll S$ with width greater than $\dfrac{\boldsymbol{\pi}}{\boldsymbol{k}}$. As $\mathcal{A}_{\boldsymbol{k}}(S) = \mathcal{W}_{\boldsymbol{k}}(S)$, there exist $C_T > 0$ and $B_T = (B_1, B_2) \in (0,\infty)^2$ such that

$$\|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\| \le C_T \boldsymbol{\alpha}!\,\Gamma(1+\boldsymbol{\alpha}/\boldsymbol{k}) B_T^{\boldsymbol{\alpha}}, \qquad \boldsymbol{z} \in T,\ \boldsymbol{\alpha} = (n,m) \in \mathbb{N}^2.$$

In view of (11), this implies that $g_n|_{T_2} \in \mathcal{W}_{k_2,B_2}(T_2)$ and

$$\|g_n|_{T_2}\|_{k_2,B_2} \le C_T \Gamma(1+n/k_1) B_1^n, \qquad n \in \mathbb{N}. \tag{12}$$

Hence $\widehat{g} = \sum_{n=0}^{\infty} g_n z_1^n$ is a k_1-Gevrey formal series with coefficients in the Banach space $\mathcal{W}_{k_2,B_2}(T_2)$. Now observe that Proposition 5.2 and the remarks following it allow identification of $f|_T \in \mathcal{W}_{\boldsymbol{k},B}(T)$ with

$$f|_T^* \in \mathcal{W}_{k_1,B_1}(T_1, \mathcal{W}_{k_2,B_2}(T_2)) \subset \mathcal{A}_{k_1}(T_1, \mathcal{W}_{k_2,B_2}(T_2)).$$

Because of (10), it is easy to see that the asymptotic expansion of $f|_T^*$ is precisely $\widehat{g}$, so that, since the width of T_1 is greater than π/k_1, we deduce that $\widehat{g}$ is k_1-summable in direction d_1 with sum $f|_T^*$. Up to isomorphism (and up to domain, inessential for analytic functions), we have recovered the "double" sum f.

Of course, a similar reasoning can be applied for the reversed ordering of the variables. So, we are led to the following definition and result.

Definition 5.3. We say a formal power series $\widehat{f} = \sum_{n,m=0}^{\infty} a_{nm} z_1^n z_2^m$ is *iteratively $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$* (in a certain order, but this will turn out to be irrelevant) if, when we write $\widehat{f} = \sum_{n=0}^{\infty} \widehat{g}_n z_1^n$, the following hold:
(i) Every $\widehat{g}_n$ is k_2-summable in direction d_2, with sum $g_n \in \mathcal{A}_{k_2}(S_2)$, where the sector $S_2 = S_2(d_2, \theta_2, \rho_2)$ does not depend on n.
(ii) There exist $T_2 = T_2(d_2, \varphi_2, r_2) \ll S_2$, with $\varphi_2 > \pi/k_2$, and $B_2(T_2) > 0$ such that $g_n \in \mathcal{W}_{k_2,B_2}(T_2)$ for every $n \in \mathbb{N}$, and the series $\widehat{g} = \sum_{n=0}^{\infty} g_n z_1^n$ is k_1-summable in direction d_1.

Proposition 5.4. *A formal power series $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ if and only if it is iteratively $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ in any order.*

Proof. The "only if" part has just been obtained. The "if" part is immediate via the isomorphism in Proposition 5.2. □

Indeed, we may give another equivalent condition for summability involving the transforms studied in the previous section.

Proposition 5.5. *A formal power series $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ if and only if $\widehat{f} \in \mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$ and $\widehat{\mathcal{B}}_{\boldsymbol{k}}\widehat{f}$, which converges, may be analytically continued to a polysector $S_{\boldsymbol{\varepsilon}} = S(\boldsymbol{d}, \boldsymbol{\varepsilon})$, obtaining a function $F \in \mathcal{A}_{(\infty,\infty)}(S_{\boldsymbol{\varepsilon}})$ such that the family $\{\boldsymbol{z}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{z})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^2}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in $S_{\boldsymbol{\varepsilon}}$, in such a way that for every $T \prec S_{\boldsymbol{\varepsilon}}$ there exist $r > 0$, $M > 0$, $C > 0$ and $A = (A_1, A_2) \in (0,\infty)^2$, all depending on T, such that for $\boldsymbol{z} \in T$ with $\|\boldsymbol{z}\|_\infty \geq r$ and for $\boldsymbol{\alpha} \in \mathbb{N}^2$ we have*

$$|F(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{z})| \leq C A^{\boldsymbol{\alpha}} \exp\big(M(|z_1|^{k_1} + |z_2|^{k_2})\big)|\boldsymbol{z}|^{\boldsymbol{\alpha}}.$$

Proof. If $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$, we know that $\widehat{f} \in \mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$, so that $\widehat{\mathcal{B}}_{\boldsymbol{k}}\widehat{f}$ converges. Also, $f := \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f}$ belongs to $\mathcal{A}_{\boldsymbol{k}}(S)$, where $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ with $\boldsymbol{\theta} > \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}}$, and $\mathrm{FA}(f) = \widehat{f}$. Taking $\bar{\boldsymbol{k}} = \boldsymbol{k}$ in Theorem 4.8, we deduce that the function $F := \mathcal{B}_{\boldsymbol{k}} f$ satisfies all the requirements.

Conversely, suppose $\widehat{f} \in \mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$ (so that $\widehat{\mathcal{B}}_{\boldsymbol{k}}\widehat{f}$ converges) and there exists the function F under the said conditions. It suffices to apply Theorem 4.1 and Remark 4.5 to conclude that $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ with sum $\mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f} = \mathcal{L}_{\boldsymbol{k}} F$. □

We are giving next some consequences of the definition. The proofs are similar to those for the corresponding properties in one variable, so they are omitted (see, for example, [2, Chapter 6]).

Proposition 5.6. *Let $\widehat{f}$ be a formal power series, $\boldsymbol{k}, \bar{\boldsymbol{k}} \in (0,\infty)^2$, $\boldsymbol{d}, \bar{\boldsymbol{d}} \in \mathbb{R}^2$.*
(i) *If $\widehat{f} \in \mathbb{C}\{\boldsymbol{z}\}$, then it is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ and $\mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f}$ agrees with the (usual) sum of $\widehat{f}$ (where both are defined).*
(ii) *If $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ and:*
(ii.a) *$\|\boldsymbol{d} - \bar{\boldsymbol{d}}\|_\infty < \varepsilon$, with $\varepsilon > 0$ small enough, then $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\bar{\boldsymbol{d}}$, and the sums agree where both are defined.*

(ii.b) $\varepsilon > 0$ *is small enough, then* $\widehat{f}$ *is* $(\boldsymbol{k} - (\varepsilon, \varepsilon))$*-summable in direction* $\bar{\boldsymbol{d}}$*, and the sums agree where both are defined.*
(ii.c) $\bar{\boldsymbol{k}} > \boldsymbol{k}$*, then* $\widehat{g} := \widehat{\mathcal{B}}_{\bar{\boldsymbol{k}}}\widehat{f}$ *is* $\widetilde{\boldsymbol{k}}$*-summable in direction* $\boldsymbol{d}$*, where* $\widetilde{k}_j^{-1} = k_j^{-1} - \bar{k}_j^{-1}$*,* $j = 1, 2$*, and* $\mathcal{S}_{\widetilde{\boldsymbol{k}},d}\widehat{g} = \mathcal{B}_{\bar{\boldsymbol{k}}}(\mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f})$.
(iii) *If* I_j *are open intervals in* $\mathbb{R}$*,* $j = 1, 2$*, and* $\widehat{f}$ *is* $\boldsymbol{k}$*-summable in direction* $\boldsymbol{d}$ *for every* $\boldsymbol{d} \in I_1 \times I_2$*, then the family* $\{\mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f}\}_{\boldsymbol{d}\in I_1\times I_2}$ *defines a unique holomorphic function.*
(iv) *If* $\widehat{f}$ *is both* $\boldsymbol{k}$*- and* $\bar{\boldsymbol{k}}$*-summable in direction* $\boldsymbol{d}$*, with* $\bar{\boldsymbol{k}} > \boldsymbol{k}$*, then* $\widehat{f}$ *is* $\bar{\boldsymbol{k}}$*-summable in all directions* $\bar{\boldsymbol{d}}$ *such that* $|d_j - \bar{d}_j| \leq \frac{\pi}{2}(k_j^{-1} - \bar{k}_j^{-1})$*,* $j = 1, 2$*, and* $\mathcal{S}_{\bar{\boldsymbol{k}},\bar{\boldsymbol{d}}}\widehat{f} = \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f}$ *where both are defined.*
(v) *If* $\boldsymbol{d} = (d_1, d_2)$ *and* $\bar{\boldsymbol{d}} = (d_1 + 2n\pi, d_2 + 2m\pi)$*, with* $n, m \in \mathbb{Z}$*, then* $\widehat{f}$ *is* $\boldsymbol{k}$*-summable in direction* $\boldsymbol{d}$ *if and only if* $\widehat{f}$ *is* $\boldsymbol{k}$*-summable in direction* $\bar{\boldsymbol{d}}$*, and in this case,* $\mathcal{S}_{\boldsymbol{k},\bar{\boldsymbol{d}}}\widehat{f}(z_1, z_2) = \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}\widehat{f}(z_1e^{-2n\pi i}, z_2e^{-2m\pi i})$ *for every* (z_1, z_2) *where both sides make sense.*

We denote by $\mathbb{C}\{\boldsymbol{z}\}_{\boldsymbol{k},\boldsymbol{d}}$ the set of $\boldsymbol{k}$-summable power series in direction $\boldsymbol{d}$, which turns out be a differential algebra with the usual operations.

Remark 5.7. Given a sector $S_1 = S(d_1, \theta_1, \rho_1)$, a disk $D_2 = D(0, \rho_2)$ and $k_1 \in (0, \infty)$, we define the space $\mathcal{A}_{(k_1,\infty)}(S_1 \times D_2)$ as consisting of the holomorphic functions $f: S_1 \times D_2 \to \mathbb{C}$ for which there exists a family

$$\mathrm{TA}(f) = \{\, h_m,\ g_n,\ a_{nm} : n, m \in \mathbb{N}\,\},$$

where h_m (resp. g_n) is a holomorphic function from S_1 (resp. D_2) to $\mathbb{C}$, and $a_{nm} \in \mathbb{C}$, $n, m \in \mathbb{N}$, such that, if we define $\mathrm{App}_{\boldsymbol{\alpha}}(f)$ as usual, then for every $T_1 \ll S_1$ and every compact $K_2 \subset D_2$ there exist $C > 0$ and $A \in (0, \infty)^2$, depending on T_1 and K_2, such that for every $\boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2$,

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})| \leq C\,\Gamma(1 + n/k_1)A^{\boldsymbol{\alpha}}|\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T_1 \times K_2.$$

We may accordingly say that a formal power series $\widehat{f}$ is (k_1, ∞)*-summable in direction* $d_1 \in \mathbb{R}$ if there exist a sector $S_1 = S(d_1, \theta_1, \rho_1)$, with $\theta_1 > \pi/k_1$, a disk $D_2 = D(0, \rho_2)$ and a function $f \in \mathcal{A}_{(k_1,\infty)}(S_1 \times D_2)$ such that $\mathrm{FA}(f) := \sum_{n,m} a_{nm}z_1^n z_2^m = \widehat{f}$.

We are again interested in giving a characterization of such series as those summable by an iterative procedure. To this end, for a complex Banach space E, a disk D and a constant $B > 0$, we introduce the vector space $\mathcal{W}_{\infty,B}(D, E)$ consisting of the holomorphic functions $f: D \to E$ such that

$$\|f\|_{\infty,B} := \sup_{z\in D,\ p\in\mathbb{N}} \frac{\|f^{(p)}(z)\|}{p!\,B^p} < +\infty.$$

$(\mathcal{W}_{\infty,B}(D, E), \|\cdot\|_{\infty,B})$ is a Banach space.

Put $\widehat{g}_n = \sum_{m=0}^{\infty} a_{nm}z_2^m$, $n \in \mathbb{N}$, and $\widehat{h}_m = \sum_{n=0}^{\infty} a_{nm}z_1^n$, $m \in \mathbb{N}$. The observations in Remark 4.4 and the proof of Proposition 5.4 provide the ideas in order to obtain the equivalence of the following three statements:
(i) $\widehat{f}$ is (k_1, ∞)-summable in direction d_1.
(ii) (a) There exists $\rho_2 > 0$ such that every $\widehat{g}_n$ converges in $D(0, \rho_2)$, with sum g_n, and (b) there exist $0 < r_2 < \rho_2$ and $A_2 > 0$ such that $g_n \in \mathcal{W}_{\infty,A_2}(D(0, r_2))$ for every n, and the series $\sum_{n=0}^{\infty} g_n z_1^n$ is k_1-summable in direction d_1.
(iii) (a) Every $\widehat{h}_m$ is k_1-summable in direction d_1, with sum $h_m \in \mathcal{A}_{k_1}(S_1)$, where the sector $S_1 = S_1(d_1, \theta_1, \rho_1)$ does not depend on m, and (b) there exist $T_1 = T_1(d_1, \varphi_1, r_1) \ll S_1$, with $\varphi_1 > \pi/k_1$, and $A_1 > 0$ such that $h_m \in \mathcal{W}_{k_1,A_1}(T_1)$ for every $m \in \mathbb{N}$, and the series $\sum_{m=0}^{\infty} h_m z_2^m$ is convergent.

In the one-variable case one considers next the differential algebra $\mathbb{C}\{\boldsymbol{z}\}_{\boldsymbol{k}}$ consisting of the formal power series which are $\boldsymbol{k}$-summable in every direction except for finitely many ones (taking into account the identification that property (v) above suggests). So, we propose to make the following

Definition 5.8. A formal power series $\widehat{f} = \sum_{n,m=0}^{\infty} a_{nm} z_1^n z_2^m$ is *$\boldsymbol{k}$-summable*, and we write $\widehat{f} \in \mathbb{C}\{\boldsymbol{z}\}_{\boldsymbol{k}}$, if there exist sets E_1 and E_2 of directions (in the z_1- and in the z_2-Riemann surface of the logarithm, respectively), both finite (after the identification suggested by (v)), such that $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d} = (d_1, d_2)$ whenever both $d_1 \notin E_1$ and $d_2 \notin E_2$.

6 A theorem of R. Gérard and Y. Sibuya revisited

Let us consider the Pfaffian system

$$(13) \qquad \begin{cases} z_1^{k_1+1} \dfrac{\partial f}{\partial z_1} &= \varphi_0(z_1, z_2) + A(z_1, z_2) f + \sum_{|\boldsymbol{\alpha}| \geq 2} f^{\boldsymbol{\alpha}} \varphi_{\boldsymbol{\alpha}}(z_1, z_2), \\ z_2^{k_2+1} \dfrac{\partial f}{\partial z_2} &= \psi_0(z_1, z_2) + B(z_1, z_2) f + \sum_{|\boldsymbol{\alpha}| \geq 2} f^{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(z_1, z_2), \end{cases}$$

where f, φ_0, $\varphi_{\boldsymbol{\alpha}}$, ψ_0 and $\psi_{\boldsymbol{\alpha}}$ are p-vectors, and A, B are $p \times p$ matrices. Suppose all data are holomorphic in $D = \{|z_1| + |z_2| < r\}$ and the series in the right members uniformly converge on compact subsets of $D \times \{\|f\| < \rho\}$. If k_1 and k_2 are positive integers, $\varphi_0(0,0) = \psi_0(0,0) = \mathbf{0}$, $A(0,0)$ and $B(0,0)$ are invertible and the system is completely integrable, then the system has a unique formal power series solution

$$\widehat{f} = \sum_{n,m \in \mathbb{N}} a_{nm} z_1^n z_2^m, \qquad a_{nm} \in \mathbb{C}^p.$$

R. Gérard and Y. Sibuya proved in [8] that $\widehat{f}$ is indeed convergent. This surprising fact contrasts with the situation for the corresponding ODE

$$z^{k+1} \frac{df}{dz} = \varphi_0(z) + A(z) f + \sum_{|\boldsymbol{\alpha}| \geq 2} f^{\boldsymbol{\alpha}} \varphi_{\boldsymbol{\alpha}}(z),$$

that, under similar conditions, has a unique formal solution that may certainly diverge, though it is always k-summable. That is why the result for $\widehat{f}$ has been reproved using different techniques by Y. Sibuya [23, 24, 25, 26]. In the last of these references, the point of view of summability is adopted, by considering one of the variables as a parameter and summing the series in the other variable "uniformly" with respect to the parameter. We shall now give still another proof based on the concept of summability that has been developed in this paper. Let us observe that, considering z_2 as a parameter, we can write the first equation in (13) as

$$(14) \qquad z_1^{k_1+1} \frac{df}{dz_1} = \varphi_0(z_1, z_2) + A(z_1, z_2) f + \sum_{|\boldsymbol{\alpha}| \geq 2} f^{\boldsymbol{\alpha}} \varphi_{\boldsymbol{\alpha}}(z_1, z_2).$$

Under the conditions stated, this system of ODE's with parameter has a unique formal power series solution which necessarily equals $\widehat{f}$. If we write

$$\widehat{f} = \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} a_{nm} z_2^m \Big) z_1^n = \sum_{n=0}^{\infty} \widehat{g}_n z_1^n,$$

it is known (see [26]) that:
(i) There exists $r_2 > 0$ such that every $\widehat{g}_n$ is convergent in $\{|z_2| < r_2\}$.
(ii) There exist $K_0 > 0$ and $B_0 > 0$ such that

$$|g_n(z_2)| \leq K_0 \Gamma(1 + n/k_1) B_0^n, \qquad |z_2| < r_2, \ n \in \mathbb{N}.$$

(iii) There exists a finite set $E_1 = \{d_1^1, d_2^1, \ldots, d_\nu^1\}$ of directions in the z_1-plane such that if $d \notin E_1$, then there exists a unique solution $F_d(z_1, z_2)$ of (14) such that:

(iii.a) F_d is holomorphic in the set V_d given by

$$z_1 \in S_d = S(d, \frac{\pi}{k_1} + \delta_d, r_d), \qquad |z_2| < r_2,$$

where $\delta_d > 0$ and $r_d > 0$ are constants.
(iii.b) There exist $K_1 > 0$ and $B_1 > 0$ such that

$$|F_d(z_1, z_2) - \sum_{n=0}^{N-1} g_n(z_2) z_1^n| \leq K_1 \Gamma(1 + N/k_1) B_1^N |z_1|^N, \quad (z_1, z_2) \in V_d,\ N \in \mathbb{N}.$$

If we take $\rho_2 < r_2$, and use (ii) and Cauchy's formula for the derivatives, we may easily deduce that $g_n \in \mathcal{W}_{\infty,B_2}(D(0,\rho_2))$, $n \in \mathbb{N}$, for suitable $B_2 = B_2(\rho_2) > 0$. Now, from (iii.b) and Cauchy's formula we obtain that the function F_d (via the usual identification) is the k_1-sum in direction d of the series $\widehat{g} = \sum_{n=0}^{\infty} g_n z_1^n$, with coefficients in $\mathcal{W}_{\infty,B_2}(D(0,\rho_2))$. Hence the series $\widehat{f}$ is (iteratively) (k_1, ∞)-summable in every direction $d_1 \notin E_1$.

Let us change the role of the variables. Consider z_1 as a parameter, and write the second equation in (13) as

$$z_2^{k_2+1} \frac{df}{dz_2} = \psi_0(z_1, z_2) + B(z_1, z_2) f + \sum_{|\boldsymbol{\alpha}| \geq 2} f^{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(z_1, z_2).$$

Reasoning as before, its unique formal power series solution, again $\widehat{f}$, turns out to be (∞, k_2)-summable in every direction d_2 except those in a certain finite set E_2. This implies that $\widehat{h}_m := \sum_{n=0}^{\infty} a_{nm} z_1^n$ converges in a disk D_1 for every m, with sum h_m. On the other hand, since $\widehat{f}$ is (k_1, ∞)-summable in every direction $d_1 \notin E_1$, we can choose a finite set of directions F_1, with $F_1 \cap E_1 = \emptyset$, in such a way that the sums $\mathcal{S}_{k_1,d} \widehat{h}_m$, as d runs over F_1, are defined in sectors T_d which cover a whole disk $\widehat{D}_1$ around 0. Of course, these sums glue together and agree with h_m. Also, the choice may be made so that there exists $A_d > 0$ such that $h_m \in \mathcal{W}_{k_1,A_d}(T_d)$ for every $m \in \mathbb{N}$, and the series $\sum_{m=0}^{\infty} h_m z_2^m$ is convergent. Taking into account that $\sup_{z_1 \in \widehat{D}_1} |h_m(z_1)| \leq \sup_{d \in F_1} \|\mathcal{S}_{k_1,d} \widehat{h}_m\|_{k_1,A_d}$, we deduce that the series $\sum_{m=0}^{\infty} h_m z_2^m$ converges in a disk D_2 when its coefficients h_m are considered in the space of holomorphic functions in $\widehat{D}_1$ (with the compact open topology). The well-known isomorphism between $\mathcal{H}(D_2, \mathcal{H}(\widehat{D}_1))$ and $\mathcal{H}(\widehat{D}_1 \times D_2)$ implies convergence of $\widehat{f}$, as desired.

7 Extension of the theory for more than two variables

When dealing with series and functions of more than two variables, the definitions and results in Sections 3, 4 and 5 may be repeated without significant change. However, in the general case, the involved definition of strongly asymptotically developable function may make some statements and proofs less pleasant. We will briefly point out these differences.

Fix $n \in \mathbb{N}$, $n > 2$, and put $N = \{1, 2, \ldots, n\}$. Notations in Section 2 are trivially adapted. We also put, for $j \in N$, $\boldsymbol{e}_j = (0, \ldots, \overset{j)}{1}, \ldots, 0)$.

Let J be a nonempty subset of N. The number of elements of J will be $\#J$. If $J = \{j_1 < j_2 < \ldots < j_k\}$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, we put $\boldsymbol{z}_J = (z_{j_1}, z_{j_2}, \ldots, z_{j_k})$. Let J and L be nonempty disjoint subsets of N. For $\boldsymbol{z}_J \in \mathbb{R}^J$ and $\boldsymbol{z}_L \in \mathbb{R}^L$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)$ represents the element of $\mathbb{R}^{J \cup L}$ satisfying $(\boldsymbol{z}_J, \boldsymbol{z}_L)_J = \boldsymbol{z}_J$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)_L = \boldsymbol{z}_L$; we also write $J^c = N - J$. In particular, we shall use these conventions for multiindices.

Finally, if $S = \prod_{j=1}^{n} S_j$ is a polysector on $\mathbb{R}^n$, then $S_J = \prod_{j \in J} S_j \subset \mathbb{R}^J$.

Let $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$ be a polysector, and $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in (0, \infty)^n$. The definitions of $\mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{k}}$ and $\mathcal{W}_{\boldsymbol{k}}(S)$ and its properties are valid.

A holomorphic function $f\colon S\to\mathbb{C}$ is said to be *Gevrey strongly asymptotically developable of order* $\boldsymbol{k}$ if there exists a family

$$\mathcal{F}=\left\{\, f_{\boldsymbol{\alpha}_J}\colon \emptyset\neq J\subset N,\ \boldsymbol{\alpha}_J\in\mathbb{N}^J\,\right\},$$

where $f_{\boldsymbol{\alpha}_J}$ is a holomorphic function from S_{J^c} to $\mathbb{C}$ when $J\neq N$, and $f_{\boldsymbol{\alpha}_J}\in\mathbb{C}$ when $J=N$, such that the following holds: if we define

$$\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})=\sum_{\emptyset\neq J\subset N}(-1)^{\#J+1}\sum_{\substack{\boldsymbol{\beta}_J\in\mathbb{N}^J\\ \boldsymbol{\beta}_J\leq\boldsymbol{\alpha}_J-\boldsymbol{1}_J}} f_{\boldsymbol{\beta}_J}(\boldsymbol{z}_{J^c})\boldsymbol{z}_J^{\boldsymbol{\beta}_J},\quad \boldsymbol{\alpha}\in\mathbb{N}^n,\ \boldsymbol{z}\in S,$$

then for every $T\ll S$ there exist $C_T>0$ and $A_T\in(0,\infty)^n$ such that

$$|f(\boldsymbol{z})-\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})|\leq C_T\,\Gamma(1+\boldsymbol{\alpha}/\boldsymbol{k})A_T^{\boldsymbol{\alpha}}|\boldsymbol{z}|^{\boldsymbol{\alpha}},\quad \boldsymbol{z}\in T,\ \boldsymbol{\alpha}\in\mathbb{N}^n.$$

$\mathcal{A}_{\boldsymbol{k}}(S)$ is the differential algebra consisting of these functions. As before, we may also consider the differential algebra of strongly asymptotically developable functions, denoted $\mathcal{A}(S)$.

Let $f\in\mathcal{A}_{\boldsymbol{k}}(S)$. The relations in (2) are now written as follows: If $\emptyset\neq J\subset N$ and $\boldsymbol{\alpha}_J\in\mathbb{N}^J$, we have

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})=\lim_{\substack{\boldsymbol{z}_J\to\boldsymbol{0}\\ \boldsymbol{z}_J\in T_J}}\frac{D^{(\boldsymbol{\alpha}_J,\boldsymbol{0}_{J^c})}f(\boldsymbol{z})}{\boldsymbol{\alpha}_J!},\quad \boldsymbol{z}_{J^c}\in S_{J^c},\tag{15}$$

for any $T_J\ll S_J$, and the limit is uniform on every $T_{J^c}\ll S_{J^c}$ whenever $J\neq N$.

$\mathrm{TA}(f)$, $\mathrm{FA}(f)=\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}f_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}$ and $\mathrm{App}_{\boldsymbol{\alpha}}(f)$ are defined, and the consistency conditions are: For every disjoint nonempty subsets J and L of N, for $\boldsymbol{\alpha}_J\in\mathbb{N}^J$ and $\boldsymbol{\alpha}_L\in\mathbb{N}^L$, and for $T_L\ll S_L$,

$$\lim_{\substack{\boldsymbol{z}_L\to\boldsymbol{0}\\ \boldsymbol{z}_L\in T_L}}\frac{D^{(\boldsymbol{\alpha}_L,\boldsymbol{0}_{(J\cup L)^c})}f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_L!}=f_{(\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)^c});\tag{16}$$

the limit is uniform on bounded proper subpolysectors of $S_{(J\cup L)^c}$ whenever $J\cup L\neq N$. Hence $f_{\boldsymbol{\alpha}_J}\in\mathcal{A}_{\boldsymbol{k}_{J^c}}(S_{J^c})$ (setting $\mathcal{A}_{\boldsymbol{k}_{N^c}}(S_{N^c})=\mathbb{C}$), and

$$\mathrm{TA}(f_{\boldsymbol{\alpha}_J})=\{\,f_{(\boldsymbol{\alpha}_J,\boldsymbol{\beta}_L)}\colon\emptyset\neq L\subset J^c,\ \boldsymbol{\beta}_L\in\mathbb{N}^L\,\}.\tag{17}$$

We have $\mathcal{W}_{\boldsymbol{k}}(S)=\mathcal{A}_{\boldsymbol{k}}(S)$. Results on the surjectivity or injectivity of the mappings FA and TA remain unchanged. For the sake of clarity, we think it is worth including the proof of Proposition 3.3 in the general case.

Proposition 7.1. *Let* $\boldsymbol{k}=(k_1,k_2,\ldots,k_n)\in(0,\infty)^n$, *and* $S=S(\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\rho})$ *be a polysector with width* $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_n)$ *such that* $\theta_j>\pi/k_j$ *for every* $j\in N$. *Then, the mapping* $\mathrm{FA}\colon\mathcal{A}_{\boldsymbol{k}}(S)\to\mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ *is injective.*

Proof. Let $f\in\mathcal{A}_{\boldsymbol{k}}(S)$ such that $\mathrm{FA}(f)=\widehat{0}$ (the null series). For every $\boldsymbol{\alpha}_{1^c}\in\mathbb{N}^{n-1}$, the function $f_{\boldsymbol{\alpha}_{1^c}}$, element of $\mathrm{TA}(f)$, belongs to $\mathcal{A}_{k_1}(S_1)$, and

$$\mathrm{TA}(f_{\boldsymbol{\alpha}_{1^c}})\equiv\mathrm{FA}(f_{\boldsymbol{\alpha}_{1^c}})=\sum_{m=0}^{\infty}f_{(m,\boldsymbol{\alpha}_{1^c})}z_1^m=\widehat{0},$$

since these coefficients constitute a subfamily of those for $\mathrm{FA}(f)$. As $\theta_1>\pi/k_1$, the injectivity of FA in the one variable case implies

$$f_{\boldsymbol{\alpha}_{1^c}}\equiv 0\quad\text{in } S_1,\ \text{for every } \boldsymbol{\alpha}_{1^c}\in\mathbb{N}^{n-1}.\tag{18}$$

Fix $z_1 \in S_1$, and for each $\boldsymbol{\alpha}_{\{1,2\}^c} \in \mathbb{N}^{n-2}$ consider the function

$$f_{\boldsymbol{\alpha}_{\{1,2\}^c}}(z_1,\cdot)\colon S_2 \to \mathbb{C},$$

which is an element of $\mathcal{A}_{k_2}(S_2)$. $f_{\boldsymbol{\alpha}_{\{1,2\}^c}}$ belongs to $\mathrm{TA}(f)$, which is consistent, so that for every $m \in \mathbb{N}$ we have

$$\lim_{z_2\to 0,\ z_2\in T_2\ll S_2} \frac{1}{m!} D^{(0,m)} f_{\boldsymbol{\alpha}_{\{1,2\}^c}}(z_1,z_2) = f_{(m_{\{2\}},\boldsymbol{\alpha}_{\{1,2\}^c})}(z_1).$$

This limit is 0 if we take (18) into account; hence $\mathrm{FA}(f_{\boldsymbol{\alpha}_{\{1,2\}^c}}(z_1,\cdot)) = \widehat{0}$, and being $\theta_2 > \pi/k_2$, we obtain that $f_{\boldsymbol{\alpha}_{\{1,2\}^c}}(z_1,\cdot) \equiv 0$. Since z_1 is arbitrary, we conclude that

$$f_{\boldsymbol{\alpha}_{\{1,2\}^c}} \equiv 0 \quad \text{in } S_1\times S_2, \text{ for every } \boldsymbol{\alpha}_{\{1,2\}^c} \in \mathbb{N}^{n-2}.$$

In the next step (necessary only if $n > 2$), we fix $(z_1, z_2) \in S_1 \times S_2$, and for each $\boldsymbol{\alpha}_{\{1,2,3\}^c} \in \mathbb{N}^{n-3}$ we prove, in a similar fashion, that the function

$$f_{\boldsymbol{\alpha}_{\{1,2,3\}^c}}(z_1,z_2,\cdot)\colon S_3 \to \mathbb{C},$$

that belongs to $\mathcal{A}_{k_3}(S_3)$, identically vanishes. So, we have that

$$f_{\boldsymbol{\alpha}_{\{1,2,3\}^c}} \equiv 0 \quad \text{in } S_1\times S_2\times S_3, \text{ for every } \boldsymbol{\alpha}_{\{1,2,3\}^c} \in \mathbb{N}^{n-3}.$$

After n steps, we conclude that $\mathrm{TA}(f)$ is the null family, and this implies $f \equiv 0$. □

Proposition 3.4 also holds in the general case. The definition of function (or family) of (uniform) exponential growth is immediately extended, as well as that of the Laplace and Borel transforms. The following are the generalizations of Theorems 4.1 and 4.8, respectively. The remarks and results after those theorems are also applicable now.

Theorem 7.2. *Let $\boldsymbol{k} = (k_1, k_2, \dots, k_n), \widetilde{\boldsymbol{k}} = (\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) \in (0,\infty)^n$, $S = S(\boldsymbol{d},\boldsymbol{\theta})$ be a polysector, and $F\colon S \to \mathbb{C}$ be a holomorphic function such that:*
(i) *$F \in \mathcal{A}_{\widetilde{\boldsymbol{k}}}(S)$, with $\mathrm{TA}(F) = \{f_{\boldsymbol{\alpha}_J}\}$;*
(ii) *The family $\{\boldsymbol{z}^{-\boldsymbol{\alpha}}(F(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{z}))\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in S, in such a way that for every $T \prec S$ there exist $r > 0$, $M > 0$, $C > 0$ and $B = (B_1, B_2, \dots, B_n) \in (0,\infty)^n$, all depending on T, such that for $\boldsymbol{z} \in T$ with $\|\boldsymbol{z}\|_\infty \geq r$ we have*

$$|F(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{z})| \leq C\Gamma(1+\boldsymbol{\alpha}/\widetilde{\boldsymbol{k}})B^{\boldsymbol{\alpha}} \exp\big(M\sum_{j=1}^{n} |z_j|^{k_j}\big)|\boldsymbol{z}|^{\boldsymbol{\alpha}}, \qquad \boldsymbol{\alpha} \in \mathbb{N}^n.$$

Then, the following hold true:
(a) *For every $\emptyset \neq J \subset N$, the family $\{f_{\boldsymbol{\alpha}_J}\}_{\boldsymbol{\alpha}_J\in\mathbb{N}^J}$ is uniformly of exponential growth at most $\boldsymbol{k}_{J^c}$ in S_{J^c}, so that one may consider $\mathcal{L}_{\boldsymbol{k}_{J^c}} f_{\boldsymbol{\alpha}_J}$, and also $\mathcal{L}_{\boldsymbol{k}} App_{\boldsymbol{\alpha}}(F)$ for every $\boldsymbol{\alpha} \in \mathbb{N}^n$.*
(b) *The function $f := \mathcal{L}_{\boldsymbol{k}} F$, defined and analytic in a polysectorial region $G = G(\boldsymbol{d}, \boldsymbol{\theta} + \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}})$, belongs to $\mathcal{A}_{\bar{\boldsymbol{k}}}(G)$, where $\bar{\boldsymbol{k}} = (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_n)$ is given by*

$$\frac{1}{\bar{k}_j} = \frac{1}{\widetilde{k}_j} + \frac{1}{k_j}, \quad j \in N,$$

and $\mathrm{TA}(f) = \{\Gamma(1+\boldsymbol{\alpha}_J/\boldsymbol{k}_J)\mathcal{L}_{\boldsymbol{k}_{J^c}} f_{\boldsymbol{\alpha}_J}\}$, so that $\mathrm{App}_{\boldsymbol{\alpha}}(f) = \mathcal{L}_{\boldsymbol{k}}\,\mathrm{App}_{\boldsymbol{\alpha}}(F)$ for every $\boldsymbol{\alpha} \in \mathbb{N}^n$.

Theorem 7.3. *Let $G = G(\boldsymbol{d},\boldsymbol{\theta})$ be a polysectorial region, $\boldsymbol{k} \in (0,\infty)^n$ be such that $\boldsymbol{\theta} > \boldsymbol{\pi}/\boldsymbol{k}$, and $f\colon G \to \mathbb{C}$ be a holomorphic function belonging to $\mathcal{A}_{\bar{\boldsymbol{k}}}(G)$, with $\bar{\boldsymbol{k}} = (\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n) \in (0,\infty)^n$ and* $\mathrm{TA}(f) = \{f_{\boldsymbol{\alpha}_J}\}$.
Let us define $\widetilde{\boldsymbol{k}} = (\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n)$ as

$$\widetilde{k}_j = \begin{cases} \Big(\dfrac{1}{\bar{k}_j} - \dfrac{1}{k_j}\Big)^{-1} & \text{if } \bar{k}_j < k_j, \\ \infty & \text{if } \bar{k}_j \geq k_j, \end{cases} \qquad j \in N.$$

Then, we have:
(a) *$F := \mathcal{B}_{\boldsymbol{k}} f \in \mathcal{A}_{\widetilde{\boldsymbol{k}}}(\widetilde{S})$, with $\widetilde{S} = S(\boldsymbol{d}, \boldsymbol{\theta} - \boldsymbol{\pi}/\boldsymbol{k})$, and $\mathrm{TA}(F) = \{\frac{\mathcal{B}_{\boldsymbol{k}_{J^c}} f_{\boldsymbol{\alpha}_J}}{\Gamma(1+\boldsymbol{\alpha}_J/\boldsymbol{k}_J)}\}$, so that $\mathrm{App}_{\boldsymbol{\alpha}}(F) = \mathcal{B}_{\boldsymbol{k}} \mathrm{App}_{\boldsymbol{\alpha}}(f)$ for every $\boldsymbol{\alpha} \in \mathbb{N}^n$.*
(b) *The family $\{\boldsymbol{u}^{-\boldsymbol{\alpha}}\big(F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})\big)\}_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ is uniformly of exponential growth at most $\boldsymbol{k}$ in $\widetilde{S}$, in such a way that for every $\widetilde{T} \prec \widetilde{S}$ there exist $r > 0$, $M > 0$, $D > 0$ and $B = (B_1, B_2, \ldots, B_n) \in (0,\infty)^n$, all depending on $\widetilde{T}$, such that for $\boldsymbol{u} \in \widetilde{T}$ with $\|\boldsymbol{u}\|_\infty \geq r$ we have*

$$|F(\boldsymbol{u}) - \mathrm{App}_{\boldsymbol{\alpha}}(F)(\boldsymbol{u})| \leq D\Gamma(1 + \boldsymbol{\alpha}/\widetilde{\boldsymbol{k}}) B^{\boldsymbol{\alpha}} \exp\big(M \sum_{j=1}^n |u_j|^{k_j}\big) |\boldsymbol{u}|^{\boldsymbol{\alpha}}, \qquad \boldsymbol{\alpha} \in \mathbb{N}^n.$$

As for summability, Definition 5.1 is valid, and similar arguments to those leading to Proposition 5.4 (see (17)) now imply

Proposition 7.4. *Let $\widehat{f} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} f_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ be a formal power series, $\boldsymbol{k} \in (0,\infty)^n$ and $\boldsymbol{d} \in \mathbb{R}^n$. The following are equivalent:*
(i) *$\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$.*
(ii) *For arbitrary $J \subset N$, $J \neq N$, we have that:*
(ii.a) *For every $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, the formal series*

$$\widehat{f}_{\boldsymbol{\alpha}_J} := \sum_{\boldsymbol{\beta}_{J^c}\in\mathbb{N}^{J^c}} f_{(\boldsymbol{\alpha}_J, \boldsymbol{\beta}_{J^c})} \boldsymbol{z}_{J^c}^{\boldsymbol{\beta}_{J^c}}$$

is $\boldsymbol{k}_{J^c}$-summable in direction $\boldsymbol{d}_{J^c}$, with sum $f_{\boldsymbol{\alpha}_J}$.
(ii.b) *There exist a (poly)sector $T_{J^c} = T(\boldsymbol{d}_{J^c}, \boldsymbol{\theta}_{J^c}, \boldsymbol{\rho}_{J^c})$, with $\boldsymbol{\theta}_{J^c} > \boldsymbol{\pi}/\boldsymbol{k}_{J^c}$, and $A_{J^c} \in (0,\infty)^{J^c}$ such that for every $\boldsymbol{\alpha}_J$, $f_{\boldsymbol{\alpha}_J}$ belongs to the Banach space $\mathcal{W}_{\boldsymbol{k}_{J^c}, A_{J^c}}(T_{J^c})$, and the series $\sum_{\boldsymbol{\alpha}_J\in\mathbb{N}^J} f_{\boldsymbol{\alpha}_J} \boldsymbol{z}_J^{\boldsymbol{\alpha}_J}$ is $\boldsymbol{k}_J$-summable in direction $\boldsymbol{d}_J$.*

The rest of Section 5 is adaptable without further comments.

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