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Attractors of magnetohydrodynamic flows in an Alfvénic state

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Abstract. We present a simplified form of the magnetohydrodynamic system which describes the evolution of a plasma where the small-scale velocity and magnetic field are aligned in the form of Alfvén waves, such as happens in several turbulent situations. Bounds on the dimension of the global attractor are found, and are shown to be an improvement of the standard ones for the full magnetohydrodynamic equations.

1. Introduction

The simplest model for describing the evolution of a plasma consists in considering it as a charged fluid, whose motion is under the influence of the Lorentz force, and where the magnetic field is determined by the induction equation, obtained by combining Faraday's law and Ohm's state relation. The resulting magnetohydrodynamic (MHD) equations for an incompressible fluid, after normalization of constants, are

$$
\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} = \nu \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla p - \nabla \left(\frac{B^2}{2} \right) + \boldsymbol{f}_1 \\
&\frac{\partial \boldsymbol{B}}{\partial t} = \eta \Delta \boldsymbol{B} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} + \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}_2 \\
&\operatorname{div} \boldsymbol{u} = 0 \\
&\operatorname{div} \boldsymbol{B} = 0
\end{aligned}
\tag{1}
$$

plus some adequate boundary and initial conditions. $\boldsymbol{u}$ stands for the fluid velocity, $\boldsymbol{B}$ is the magnetic field, p is the kinetic pressure, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are possible forcing terms, ν is the fluid viscosity and η the resistivity, also called magnetic diffusivity.

When $\boldsymbol{B} = \mathbf{0}$ we obtain the classical Navier–Stokes equations. A well known suggestion [1–3] is that the onset of hydrodynamic turbulence occurs when the solutions approach a topologically complicated attractor. Let us remember that a global attractor is a compact subset $\mathcal{A}$ of the underlying space (in this case the set of square-integrable functions in the domain under consideration) such that the evolution of any bounded set in this space will take it arbitrarily close to $\mathcal{A}$. The existence of global attractors has been proved for the two-dimensional incompressible Navier–Stokes and MHD equations (see e.g. [4]). The situation is different in the three-dimensional case: it is not known if classical solutions exist for all time, although they certainly do so at least up to some time depending on the initial condition. Although in principle all the information concerning the evolution of the system is contained in (1), the enormous analytic and numerical difficulties of integrating the equation for large times force us to use a more heuristic type of argument to explain observable phenomena. In particular in turbulent plasmas, where sharp space and time variations are common, a number of features are present which provide additional information about the original system. One of them is the Alfvén effect. The small-scale components of velocity and field, $\boldsymbol{u}_d$ and $\boldsymbol{B}_d$, tend to be aligned in the form of Alfvén waves ($\boldsymbol{u}_d = \pm\boldsymbol{B}_d$), which account for the null convective effect of these modes: $\boldsymbol{u}_d \cdot \nabla\boldsymbol{u}_d - \boldsymbol{B}_d \cdot \nabla\boldsymbol{B}_d = \boldsymbol{0}$, $\boldsymbol{u}_d \cdot \nabla\boldsymbol{B}_d - \boldsymbol{B}_d \cdot \nabla\boldsymbol{u}_d = \boldsymbol{0}$. Several explanations of this fact exist: one is that these small-scale fields tend to minimize the energy

$$\int u^2 + B^2 \,\mathrm{d}V$$

while maintaining the so-called cross-helicity

$$\int \boldsymbol{u} \cdot \boldsymbol{B} \,\mathrm{d}V.$$

The extremum condition gives $\boldsymbol{u} = \pm\boldsymbol{B}$ [5]. Also, eddies will only interact when two counter-propagating Alfvén waves on neighbouring field lines intersect, transferring energy to lower modes [6]. It is also known that if we have a static equilibrium and consider small variations $\boldsymbol{v}$, $\boldsymbol{b}$ of it, these are governed by the linearized MHD equations [7–9]

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} &= \nu\Delta\boldsymbol{v} + \boldsymbol{B}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{B} - \nabla(p_1 + \boldsymbol{B}\cdot\boldsymbol{b}) \\ \frac{\partial \boldsymbol{b}}{\partial t} &= \eta\Delta\boldsymbol{b} - \boldsymbol{v}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\boldsymbol{v}.\end{aligned} \tag{2}$$

In the ideal case ($\eta = \nu = 0$) the only possible solutions correspond to (Alfvén) waves propagating along the field lines of $\boldsymbol{B}$, normal to them, and such that $\boldsymbol{v} = \pm\boldsymbol{b}$. Therefore, if we consider the large-scale field as quasi-static as compared with the small-scale one, and larger than it in magnitude, the Alfvénic approximation is valid. A consequence of this is *equipartition*: $v^2 = b^2$, i.e. the kinetic and magnetic energies are equal for these modes. This assumption, admittedly not rigorously justified, seems to hold in most studied plasmas [10–12] and we will accept it as part of our hypotheses.

As announced, we will substitute the standard MHD equations by a simpler model which will approximate the real one for plasmas where the Alfvén effect is in force and the energy of the small-scale components is much smaller than the total one. To state the assumptions of the modified MHD model, we will concentrate on a periodic box $I = [0, 2\pi]^3$ because the eigenfunctions of the Laplacian are precisely the complex exponentials occurring in the classical Fourier mode turbulence analysis, and which moreover are closed for the product of functions. The following space is defined:

$$H = \left\{ \boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B}) \in L^2(I)^6 : \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0, \boldsymbol{w} \text{ periodic}, \int_I \boldsymbol{w}\,\mathrm{d}V = \boldsymbol{0} \right\}. \tag{3}$$

The condition $\int_I \boldsymbol{w}\,\mathrm{d}V = \boldsymbol{0}$ is mathematically convenient to later apply the divergence theorem. If it holds for all time for the forcing terms $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ and for the initial condition, $\boldsymbol{w}(0)$, it holds for all time, both for the system (1) and for the modified one we will define later.

Let us remember that the Sobolev space $H^m(I)$ is formed by the functions whose m first partial derivatives are square-integrable in I, and that its norm is given by

$$\|u\|^2_{H^m(I)} = \sum_{|\alpha|\leqslant m} |D^\alpha u|^2_{L^2(I)}.$$

We will use mostly $m = 0$ (i.e. the space $L^2(I)$) whose norm will be abbreviated to $|\ |$, and $m = 1$, writing $\|\ \|$ instead of $\|\ \|_1$. V will be the subspace $H \cap H^1(I)^6$. Since in principle the functions in H do not need to be even continuous, it must be assumed that the conditions $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0$ are taken in the sense of distributions and the periodicity means that the appropriate *traces* at the boundaries coincide. Those traces are known to exist within a well-defined function space. The projection of any gradient, such as $\nabla(p + B^2/2)$, in H vanishes.

We will divide the set of Fourier modes in two classes: the mean one, where the MHD equations hold without further constraint, and the diffusive range, where components are in an Alfvénic state. The limit k_d between these classes in fact depends on the large-scale field through the Alfvén velocity and the energy dissipation. Hence the modified system with a constant k_d will describe correctly the evolution of the plasma when we are within an invariant subset of trajectories such that the Alfvén velocity and the energy dissipation are roughly constant there. Such a thing really occurs in many turbulent situations, which are the ones we consider; thus, although we will study this system within whole function spaces, we will always obtain our conclusions from restrictions to invariant turbulent states. We must remember that turbulence is by no means a general feature of real plasmas; for instance, a large mean magnetic field will tend to suppress it. Every magnitude $\boldsymbol{f}$ has two obvious components: the mean $\boldsymbol{f}_m$ and the dissipative one $\boldsymbol{f}_d$, corresponding to the Fourier sum of the modes within each of these ranges. We will assume that $(\boldsymbol{u}_d; \boldsymbol{B}_d)$ are in an Alfvén state, in the sense that the nonlinear interaction

$$\begin{aligned} &\boldsymbol{u}_d \cdot \nabla \boldsymbol{u}_d - \boldsymbol{B}_d \cdot \nabla \boldsymbol{B}_d \\ &\boldsymbol{u}_d \cdot \nabla \boldsymbol{B}_d - \boldsymbol{B}_d \cdot \nabla \boldsymbol{u}_d \end{aligned} \tag{4}$$

vanishes. Also, the energy in the dissipative range is much smaller than the total one: $|\boldsymbol{u}_d| \ll |\boldsymbol{u}_m|$, $|\boldsymbol{B}_d| \ll |\boldsymbol{B}_m|$, so we will take all the terms of form $\boldsymbol{f}_d \cdot \nabla \boldsymbol{g}_m$ as zero. By its very nature of rapid variation, such a thing cannot be asserted of the terms in $\boldsymbol{f}_m \cdot \nabla \boldsymbol{g}_d$, so our system is really infinite dimensional and not a mere truncation of the original one. Hence our turbulent MHD system becomes

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} = \nu \Delta \boldsymbol{u} - \boldsymbol{u}_m \cdot \nabla \boldsymbol{u} + \boldsymbol{B}_m \cdot \nabla \boldsymbol{B} - \nabla \left(p + \frac{B^2}{2} \right) + \boldsymbol{f}_1(t) \\ &\frac{\partial \boldsymbol{B}}{\partial t} = \eta \Delta \boldsymbol{B} - \boldsymbol{u}_m \cdot \nabla \boldsymbol{B} + \boldsymbol{B}_m \cdot \nabla \boldsymbol{u} + \boldsymbol{f}_2(t) \\ &\operatorname{div} \boldsymbol{u} = 0 \\ &\operatorname{div} \boldsymbol{B} = 0. \end{aligned} \tag{5}$$

Notice that $\boldsymbol{u}_m$ and $\boldsymbol{B}_m$ also have null divergence and are periodic. Take $\lambda = \min\{\nu, \eta\} > 0$, and let $D = (\nu\Delta; \eta\Delta)$. $\boldsymbol{F} = (\boldsymbol{f}_1; \boldsymbol{f}_2)$ is a forcing term accounting for possible external influences.

2. Attractors of the system

Let us define the trilinear form b

$$b(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \int_I (\boldsymbol{f} \cdot \nabla \boldsymbol{g}) \boldsymbol{h} = \int_I \sum_{i,j} f_i \frac{\partial g_j}{\partial x_i} h_j \tag{6}$$

which allows us to define $C : V \times V \to V'$ by

$$\begin{aligned} (C[(\boldsymbol{u}_1; \boldsymbol{B}_1), (\boldsymbol{u}_2; \boldsymbol{B}_2)], (\boldsymbol{u}_3; \boldsymbol{B}_3)) &= b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) - b(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{u}_3) \\ &\quad + b(\boldsymbol{u}_1, \boldsymbol{B}_2, \boldsymbol{B}_3) - b(\boldsymbol{B}_1, \boldsymbol{u}_2, \boldsymbol{B}_3). \end{aligned} \tag{7}$$

After projecting in the subspace of $L^2(I)$ functions with null divergence, the MHD system may be written

$$\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} = D\boldsymbol{w} - C(\boldsymbol{w}, \boldsymbol{w}) + \boldsymbol{F} \\ &\boldsymbol{w}(0) = \boldsymbol{w}_0. \end{aligned} \tag{8}$$

The following facts are now easy to prove.

The bilinear form

$$\begin{aligned} &C_m : V \times V \to V' \\ &C_m(\boldsymbol{w}, \boldsymbol{w}') = C(\boldsymbol{w}_m, \boldsymbol{w}') \end{aligned} \tag{9}$$

satisfies

$$(C_m(\boldsymbol{w}_1, \boldsymbol{w}_2), \boldsymbol{w}_2) = 0 \tag{10}$$

$$|(C_m(\boldsymbol{w}_1, \boldsymbol{w}_2), \boldsymbol{w}_3)| \leqslant M|\boldsymbol{w}_1|\|\boldsymbol{w}_2\||\boldsymbol{w}_3|. \tag{11}$$

The first identity is a consequence of the divergence theorem and the fact that all the functions are periodic, so the boundary integrals vanish. The bound follows because in the finite-dimensional space of functions with Fourier modes bounded by k_d, the maximum and $|\ |$ are equivalent norms. As a matter of fact, it would be desirable to keep the terms $\boldsymbol{f}_d \cdot \nabla \boldsymbol{g}_m$, thus making strict use of the Alfvén hypothesis: unfortunately, this will prove mathematically awkward later. In particular, condition (10) would not hold, although the bounds in (11) would still be valid. Rather than further complicate the mathematics, we will ignore this small term. By the same reason we will assume that the forcing term $\boldsymbol{F}$ is independent of time. In this way the solutions $\boldsymbol{w}(t)$ will define a semigroup of operators $S(t) : \boldsymbol{w}_0 \to \boldsymbol{w}(t)$. Now the modified MHD system becomes

$$\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} = D\boldsymbol{w} - C_m(\boldsymbol{w}, \boldsymbol{w}) + \boldsymbol{F} \\ &\boldsymbol{w}(0) = \boldsymbol{w}_0. \end{aligned} \tag{12}$$

A consequence of inequality (11) in the nonlinear terms of system (12) is that it has a unique solution

$$\boldsymbol{w} \in C([0, \infty), H) \cap L^2([0, T], V) \tag{13}$$

for all time $T > 0$ if $\boldsymbol{w}_0 \in H$, and

$$\boldsymbol{w} \in C([0, \infty), V) \cap L^2([0, T], H^2(I)^6) \tag{14}$$

for every $T > 0$ if $\boldsymbol{w}_0 \in V$. Moreover, there exists a maximal attractor which is compact and connected in H. This result is a consequence of theorem 3.1 in [4, pp 113–16]. Essentially the proof relies on the fact that there is an absorbing set (where all the solutions eventually lie) which is bounded not only in H, but also in V, and therefore relatively compact in H.

In order to bound the dimension of the attractor of this system, we recall some basic notation and results from [4]. Let us write (12) as

$$\frac{\partial \boldsymbol{w}}{\partial t} = \boldsymbol{G}(\boldsymbol{w}) + \boldsymbol{F}. \tag{15}$$

Let $\boldsymbol{w}(t)$ be the solution with $\boldsymbol{w}(0) = \boldsymbol{w}_0$, and consider the linearized equation

$$\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{G}'(\boldsymbol{w}(t))\boldsymbol{U} \\ &\boldsymbol{U}(0) = \xi. \end{aligned} \tag{16}$$

Let $\phi_1(t), \ldots, \phi_m(t)$ be an orthonormal base of the space of solutions to system (16) associated to the m-dimensional space of initial conditions $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m$. Functions ϕ_j may be taken to depend continuously on t. Take

$$\Gamma_m(t, \boldsymbol{w}_0) = \sum_{j=1}^{m} (G'(\boldsymbol{w}(t))\phi_j(t), \phi_j(t)) \tag{17}$$

(the internal product in the space H, i.e. the usual L^2 product) and let

$$
\begin{aligned}
q_m(t) &= \sup_{w_0 \in X} \sup_{\xi_j \in H, |\xi_j| \leqslant 1} \left(\frac{1}{t} \int_0^t \Gamma_m(\tau, \boldsymbol{w}_0)\, \mathrm{d}\tau \right) \\
q_m &= \limsup_{t \to \infty} q_m(t)
\end{aligned}
\tag{18}
$$

where X is some bounded invariant subset of the semigroup. There is an additional technical condition: the semigroup $S(t)$ associated must be *uniformly differentiable* in the attractor $\mathcal{A}$ [4, p 282 ss]. This follows in our case by a general result concerning a class of equations which include (12) [4, p 373 ss]. If for some m, $q_m < 0$, the Hausdorff dimension of the attractor of the functions within X is bounded by m, and the fractal dimension by $2m$.

The linearized equation at a point $\boldsymbol{w}$ is

$$
\frac{\partial \boldsymbol{U}}{\partial t} = D\boldsymbol{U} + C_m(\boldsymbol{w}, \boldsymbol{U}) + C_m(\boldsymbol{U}, \boldsymbol{w}). \tag{19}
$$

Obviously

$$
(D\boldsymbol{U}, \boldsymbol{U}) \leqslant -\lambda \|\boldsymbol{U}\|. \tag{20}
$$

On the other hand, $(C_m(\boldsymbol{w}, \boldsymbol{U}), \boldsymbol{U}) = 0$, and if we denote $\boldsymbol{U} = (\boldsymbol{v}; \boldsymbol{b})$, $\boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B})$,

$$
(C_m(\boldsymbol{U}, \boldsymbol{w}), \boldsymbol{U}) = \int_I (\boldsymbol{v}_m \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{v} - (\boldsymbol{b}_m \cdot \nabla \boldsymbol{B}) \cdot \boldsymbol{v} + (\boldsymbol{v}_m \cdot \nabla \boldsymbol{B}) \cdot \boldsymbol{b} - (\boldsymbol{b}_m \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{b}. \tag{21}
$$

Consider the first summand in the integral: the rest are analogous. It is bounded by $\|\boldsymbol{v}_m\|_\infty |\nabla \boldsymbol{v}| |\boldsymbol{u}|$. If $\boldsymbol{v}_m$ is the sum of the harmonics up to k_d,

$$
\|\boldsymbol{v}_m\|_\infty \leqslant \left(\sum_{|\boldsymbol{k}| \leqslant k_d} |\hat{v}(\boldsymbol{k})|^2 \right)^{1/2} (2k_d)^{n/2} \leqslant |\boldsymbol{v}| (2k_d)^{n/2} \tag{22}
$$

where n is the space dimension. Since $|\nabla \boldsymbol{v}| \leqslant \|\boldsymbol{v}\|$, adding all the terms and applying Cauchy–Schwarz's inequality, we get

$$
|(C_m(\boldsymbol{U}, \boldsymbol{w}), \boldsymbol{U})| \leqslant (2k_d)^{n/2} |\boldsymbol{U}|^2 \|\boldsymbol{w}\|. \tag{23}
$$

Hence

$$
\Gamma_m(t, \boldsymbol{w}_0) \leqslant \sum_{j=1}^{m} (-\lambda \|\boldsymbol{\phi}_j\|^2 + (2k_d)^{n/2} |\boldsymbol{\phi}_j|^2 \|\boldsymbol{w}(t)\|). \tag{24}
$$

First, $|\boldsymbol{\phi}_j| = 1$. Second, a generalization of the Lieb–Thirring inequality [4, p 466 ss] shows that there exists a positive constant δ, independent of λ and $\boldsymbol{\phi}_j$ such that

$$
\sum_{j=1}^{m} \|\boldsymbol{\phi}_j\|^2 \geqslant \delta m^{(2/n)+1}. \tag{25}
$$

Finally, we use an expression of the dissipative frequency k_d arising from scaling analysis [5, p 201]:

$$
k_d = \left(\frac{\mathcal{E}}{\nu^2 v_A} \right)^{1/3} \tag{26}
$$

where it is assumed that ν and η are of the same order, so that we may take $\nu = \lambda$. v_A is the Alfvén velocity, proportional in our conservative case to the mean magnetic field, which we assume constant in X. Finally, $\mathcal{E}$ is the energy dissipation flux, defined as the *ensemble average*

$$
\mathcal{E} = \lambda \langle \nabla \boldsymbol{w} \rangle^2 \tag{27}
$$

which may be interpreted as

$$\mathcal{E} = \lambda \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|^2\,\mathrm{d}s. \tag{28}$$

This will not depend on the trajectory $\boldsymbol{w}(t)$ if the subset X is formed by ergodic trajectories: anyway, $\mathcal{E}$ is bounded by

$$\lambda \limsup_{t\to\infty} \sup_{w_0\in X} \frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|^2\,\mathrm{d}s. \tag{29}$$

Integrating (24) and using Cauchy–Schwarz's inequality,

$$\frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|\,\mathrm{d}s \leqslant \left(\frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|^2\,\mathrm{d}s\right)^{1/2} \tag{30}$$

after substituting $\mathcal{E}$ by its bound, we get

$$q_m \leqslant -\delta\lambda m^{(2/n)+1} + 2^{n/2}(\lambda v_A)^{-n/6} m \left(\limsup_{t\to\infty} \sup_{w_0\in X} \frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|^2\,\mathrm{d}s\right)^{(n/6)+(1/2)}. \tag{31}$$

We see that the key term is the mean integral of $\|\boldsymbol{w}\|^2$. There is a bound of it, obtained by standard energy inequalities: it may be seen in [4, pp 334–5] for dimension two, but it holds in any dimension given properties (10) and (11). It comes as

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\boldsymbol{w}(s)\|^2\,\mathrm{d}s \leqslant C\lambda^{-2}|\boldsymbol{F}|^2 \tag{32}$$

where C is a constant not depending on λ. To get $q_m < 0$, we therefore need

$$m \geqslant \left(\frac{2^{n^2/4} C^{(n^2/12)+(n/4)}}{\delta^{n/2} v_A^{n^2/12}}\right) \lambda^{-(n^2/4)-n} |\boldsymbol{F}|^{(n^2/6)+(n^2/2)} \tag{33}$$

so that the Hausdorff and fractal dimensions of the attractor may blow up as resistivity and viscosity decrease at most like $\lambda^{-21/4}$ for three-dimensional plasmas and λ^{-3} for two-dimensional ones. For the full two-dimensional MHD system, present bounds obtain a scaling of λ^{-4}, which highlights the importance of the Alfvén effect in simplifying the turbulent behaviour of the plasma. Obviously there are no known bounds for three-dimensional magnetohydrodynamics.

The practical meaning of those estimates is that they represent in a sense the number of degrees of freedom in the long-term evolution of the flow. In our case, for an m such as in (33), the attractor can be parametrized by $N = 2m + 1$ variables, in the sense that there exists an embedding of $\mathcal{A}$ in $\mathbb{R}^N$. If we take $N = 4m + 1$, we may even take this embedding to be an orthogonal projection whose inverse is Hölder continuous [13, 14].

3. Conclusions

We have studied a modified MHD system intended to take into account the Alfvén effect within the dissipative modes in a turbulent plasma. This system will correctly describe the evolution of velocity and magnetic field as long as the convective action of the small-scale components of these magnitudes vanishes. This system possesses a global attractor whose Hausdorff and fractal dimensions have at most an order of the minimum of the magnetic and kinetic viscosities elevated to $-\frac{21}{4}$, in the three-dimensional case, and to -3 in the two-dimensional one, which reflect the simplifying influence of the Alfvén effect on a turbulent plasma.

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