Filtering of Multidimensional Semiperiodic Signals

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***Abstract*—Filters on the space of multidimensional semiperiodic signals are studied. We show that the filters on these signals whose transfer function is reasonably smooth satisfy many additional properties. As an application, we prove that semiperiodic solutions of a class of difference equations can be found under a condition weaker than those previously known.**

***Index Terms*—Filtering, Fourier transform, semiperiodic signal.**

I. Introduction

Multidimensional discrete signals have many applications to image scanning, computer-aided tomography, geophysics, design of passive sonar arrays, noise removal, etc. [4], [11]. Among them, periodic signals have a special interest because of the basic role played by the discrete Fourier transform in signal theory and the computational speed provided by the fast Fourier transform. In our study of the engineering literature, we have noticed the lack of a solid theoretical background to get a consistent definition of basic tools such as filtering, moment expansion, etc. Our correspondence discusses the application of ideas from theoretical harmonic analysis to the study of filters on the space of multidimensional semiperiodic signals.

The space of semiperiodic sequences is the completion of the space of periodic sequences with the supremum norm. Berg [2] proves that this space is the Banach algebra $\mathcal{C}(\Delta)$ of continuous functions in a compact group Δ. Later, Núñez in [13] and [14] obtains a similar result, giving more information about the group Δ. The generalization of these results to the multidimensional case will allow us to prove that filters on k-dimensional semiperiodic signals may be characterized in terms of Borel measures on a certain compact group Δ_k.

We will describe briefly the main mathematical results on the space of semiperiodic signals. Complete proofs may be found in [1], [5]–[7], [9], and [12].

A *discrete multidimensional signal* is a complex function defined on $\mathbb{Z}^k$. It is denoted by $\{x(\boldsymbol{n})\}_{n\in\mathbb{Z}^k}$ or $\underset{\sim}{x}$.

A discrete signal $\{x(\boldsymbol{n})\}_{n\in\mathbb{Z}^k}$ is said to be *periodic* if there exists a regular matrix $\boldsymbol{T}=(t_{ij})$ of order k with coefficients in $\mathbb{Z}$ such that

$$x(\boldsymbol{n}+\boldsymbol{T}\boldsymbol{m})=x(\boldsymbol{n}), \qquad \text{for } \boldsymbol{n},\, \boldsymbol{m}\in\mathbb{Z}^k.$$

The matrix $\boldsymbol{T}$ is called a *period* of $\underset{\sim}{x}$. The set $p(k)$ of the periodic discrete signals defined on $\mathbb{Z}^k$ is not complete with the supremum norm.

We define in $\mathcal{GL}(k,\,\mathbb{Z})$, which is the multiplicative semigroup of regular matrices of order k with coefficients in $\mathbb{Z}$, the relation $\boldsymbol{S}\prec\boldsymbol{T}$ if there exists $\boldsymbol{P}\in\mathcal{GL}(k,\,\mathbb{Z})$ such that $\boldsymbol{T}=\boldsymbol{S}\boldsymbol{P}$. This is a preorder relation such that for $\boldsymbol{S},\boldsymbol{T}$, there exists $\boldsymbol{R}$ with $\boldsymbol{S},\boldsymbol{T}\prec\boldsymbol{R}$. In other words, $\mathcal{GL}(k,\,\mathbb{Z})$ is a directed set.

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A discrete signal $\{x(\boldsymbol{n})\}_{n\in\mathbb{Z}^k}$ is called *semiperiodic* if for every real number $\varepsilon>0$ there exists a matrix $\boldsymbol{T}$ of $\mathcal{GL}(k,\,\mathbb{Z})$ satisfying

$$|x(\boldsymbol{n})-x(\boldsymbol{n}+\boldsymbol{T}\boldsymbol{m})|<\varepsilon, \qquad \text{for } \boldsymbol{n},\, \boldsymbol{m}\in\mathbb{Z}^k.$$

The space $sp(k)$ of semiperiodic signals on $\mathbb{Z}^k$ is the completion of $p(k)$ with respect to the supremum norm, i.e., given a sequence of periodic functions that is Cauchy with respect to the supremum norm, it converges to a semiperiodic signal. Moreover, given a fixed semiperiodic signal $\underset{\sim}{x}$, there exits a sequence of periodic signals that converges in the uniform limit to $\underset{\sim}{x}$ (see [9, pp. 191–193]).

Semiperiodic signals are precisely the uniformly continuous functions on $\mathbb{Z}^k$ with the topology $\mathcal{T}_k$ that has $\{V_{n,\,T}\colon \boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z})\}$ as a fundamental system of open neighborhoods of $\boldsymbol{n}$, where

$$V_{n,\,T}=\{\boldsymbol{n}+\boldsymbol{T}\boldsymbol{m}\colon \boldsymbol{m}\in\mathbb{Z}^k\}, \qquad \boldsymbol{n}\in\mathbb{Z}^k,\ \boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z}).$$

This topology $\mathcal{T}_k$ is compatible with the group structure on $\mathbb{Z}^k$.

If $\boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z})$, the set $\boldsymbol{T}\mathbb{Z}^k$ is a subgroup of the additive group $\mathbb{Z}^k$. The corresponding quotient set $\mathbb{Z}^k/\boldsymbol{T}\mathbb{Z}^k$ is finite, and $\mathrm{card}(\mathbb{Z}^k/\boldsymbol{T}\mathbb{Z}^k)=|\det(\boldsymbol{T})|$ [3, pp. 9–14].

Following the idea given in [13] and [14] in the case $k=1$, we may obtain a representation of the completion of $(\mathbb{Z}^k,\,\mathcal{T}_k)$ as the projective limit of the inverse mapping system given by the projections of $\mathbb{Z}^k/\boldsymbol{T}\mathbb{Z}^k$ onto $\mathbb{Z}^k/\boldsymbol{S}\mathbb{Z}^k$, $\boldsymbol{S}\prec\boldsymbol{T}$. This group, which is denoted by Δ_k, is compact, and it is possible to identify the space of the semiperiodic signals with the space of continuous functions on Δ_k. In particular, every discrete periodic signal $\underset{\sim}{x}$ of period $\boldsymbol{T}$ is associated with its unique continuous extension on Δ_k

$$\sum_{\boldsymbol{n}\in\mathcal{P}(\boldsymbol{T})} x(\boldsymbol{n})\chi_{\overline{V}_{n,\,T}}$$

where $\mathcal{P}(\boldsymbol{T})=\{\sum_{j=1}^{k}\lambda_j\boldsymbol{t}_j\colon \lambda_j\in[0,\,1)\}\cap\mathbb{Z}^k$, and $\chi_{\overline{V}_{n,\,T}}$ is the characteristic function of the closure of $V_{n,\,T}$ in Δ_k. We remark that $\mathcal{P}(\boldsymbol{T})$ contains a unique representative of each equivalence class of $\mathbb{Z}^k/\boldsymbol{T}\mathbb{Z}^k$.

II. Filters of Semiperiodic Signals

If G is a conmutative compact group with dual group $\hat{G}$, a *filter* (which is usually called *multiplier* in mathematical terminology) on $\mathcal{C}(G)$ is a continuous operator ϕ on $\mathcal{C}(G)$, which is shift-invariant [i.e., $\tau_a(\phi g)=\phi(\tau_a g)$ for $a\in G$, where $\tau_a g(u)=g(u-a)$].

A classical theorem in this theory, whose proof can be found in [6], [10], and [12], states that each filter ϕ on $\mathcal{C}(G)$ can be associated with a Borel measure μ on G such that

$$\phi g=\mu * g, \qquad \text{for } g\in\mathcal{C}(G).$$

The Fourier–Stieltjes transform of μ, $f=\hat{\mu}$, which is defined on $\hat{G}$, satisfies

$$(\phi g)\hat{}\,(\gamma)=f(\gamma)\hat{g}(\gamma), \qquad \text{for } g\in\mathcal{C}(G),\ \gamma\in\hat{G}.$$

In signal theory, the function f is usually called the *transfer function* of the filter.

Given this definition, we note that there are as many filters as there are Borel measures on G. From a practical point of view, certain filters have a special significance. Since $f(\gamma)$ represents the filter's answer to a signal with frequency γ, for any implementable filter, it is reasonable that the output of different signals keeps a certain uniformity with respect to the frequencies; in other words, the function $f(\gamma)$ must satisfy some properties of regularity with respect to γ. We prove that a weak condition, such as the continuity of f, implies that the filter is of a specific type and that the function f belongs to the algebra $A(\mathbb{T}^k)$ of the continuous functions on $\mathbb{T}^k$, whose Fourier series is absolutely summable.

Theorem: Let ϕ: $sp(k) \to sp(k)$ be a linear map. The following statements are equivalent.

1) ϕ is a filter associated with a measure concentrated on $\mathbb{Z}^k$.
2) There exists an absolutely summable family $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ of complex numbers satisfying
$$\phi x = \left\{\sum_{\boldsymbol{m}\in\mathbb{Z}^k} a_{\boldsymbol{n}-\boldsymbol{m}} x(\boldsymbol{m})\right\}_{\boldsymbol{n}\in\mathbb{Z}^k}, \qquad \text{for } x \in sp(k).$$
3) There exists a continuous function f on the k-dimensional torus $\mathbb{T}^k$ such that
$$(\phi x)\hat{}\,(\boldsymbol{z}) = f(\boldsymbol{z})\hat{x}(\boldsymbol{z}), \qquad \text{for } \boldsymbol{z} \in \Gamma_k.$$

Proof: Suppose that ϕ is a filter associated with a measure $\mu \in M(\Delta_k)$ concentrated on $\mathbb{Z}^k$. There exists an absolutely summable family $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ of complex numbers such that
$$\mu = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}}\delta_{\boldsymbol{n}}$$
where $\delta_{\boldsymbol{n}}$ stands for the Dirac measure centered at the point $\boldsymbol{n}$. Then, for each $\boldsymbol{n} \in \mathbb{Z}^k$
$$\begin{aligned}(\phi x)(\boldsymbol{n}) &= \mu * x(\boldsymbol{n})\\ &= \int_{\Delta_k} x(\boldsymbol{n}-u)\,d\mu(u)\\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^k} a_{\boldsymbol{m}} x(\boldsymbol{n}-\boldsymbol{m})\\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^k} a_{\boldsymbol{n}-\boldsymbol{m}} x(\boldsymbol{m})\end{aligned}$$
which shows that 1) implies 2).

If 2) holds, it makes sense to define the measure $\mu \in M(\Delta_k)$ by $\mu = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}}\delta_{\boldsymbol{n}}$. For $x \in sp(k)$ and for $\boldsymbol{n} \in \mathbb{Z}^k$, we have that
$$(\phi x)(\boldsymbol{n}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} a_{\boldsymbol{n}-\boldsymbol{m}} x(\boldsymbol{m}) = \mu * x(\boldsymbol{n})$$
and the equivalence of 1) and 2) is proved.

Now, we prove that 1) implies 3). The group $\Gamma_k = \{(z_1, \cdots, z_k) \in \mathbb{C}^k\colon z_j \text{ is a root of the unity}\}$, with the discrete topology is the dual group of Δ_k [8]. The function z defined by
$$z(\boldsymbol{n}) = \boldsymbol{z}^{\boldsymbol{n}} = z_1^{n_1}\cdots z_k^{n_k}, \qquad \boldsymbol{n} \in \mathbb{Z}^k \tag{1}$$
gives $\boldsymbol{z} \in \Gamma_k$ as a character on Δ_k. If $\mu = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}}\delta_{\boldsymbol{n}}$, its Fourier–Stieltjes transform is given by
$$\hat{\mu}(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}}\boldsymbol{z}^{-\boldsymbol{n}}, \qquad \text{for } \boldsymbol{z} \in \Gamma_k.$$
Since the family $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ is absolutely summable, $\hat{\mu}$ can be extended to a continuous function f on $\mathbb{T}^k$
$$f(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}}\boldsymbol{z}^{-\boldsymbol{n}}.$$

The proof that 3) implies 1) is based on the following lemma.

Lemma: Let f be the transfer function of a filter. If f can be extended to a Riemann integrable function on $\mathbb{T}^k$ and the measure $\mu \in M(\Delta_k)$ such that $\hat{\mu} = f$ satisfies $\mu(\{\boldsymbol{n}\}) = 0$, for $\boldsymbol{n} \in \mathbb{Z}^k$, then
$$f = 0 \qquad \text{almost everywhere in } \mathbb{T}^k.$$

Proof of the Lemma: Let $\{\mu_{\boldsymbol{T}}\}_{\boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z})}$ be the family of periodic signals associated with μ by
$$\mu_{\boldsymbol{T}}(\boldsymbol{n}) = \mu(\overline{V}_{n,\,T}).$$

If $\boldsymbol{T}$ is a matrix of $\mathcal{GL}(k, \mathbb{Z})$, we denote by $\Gamma^{\boldsymbol{T}}$ the subgroup of Γ_k given by the elements $\boldsymbol{z} = (z_1, \cdots, z_k) \in \Gamma_k$ such that the function z defined on $\mathbb{Z}^k$ by (1) is $\boldsymbol{T}$-periodic. If x is a periodic signal of period $\boldsymbol{T}$, its Fourier transform is given by the expression
$$\hat{x}(\boldsymbol{z}) = \frac{1}{\det \boldsymbol{T}} \sum_{\boldsymbol{n}\in\mathcal{P}(\boldsymbol{T})} \boldsymbol{z}^{-\boldsymbol{n}} x(\boldsymbol{n}), \qquad \text{if } \boldsymbol{z} \in \Gamma^{\boldsymbol{T}}$$
and the Fourier–Stieltjes transform of μ satisfies
$$\hat{\mu}(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathcal{P}(\boldsymbol{T})} \boldsymbol{z}^{-\boldsymbol{n}}\mu(\overline{V}_{n,\,T}), \qquad \text{if } \boldsymbol{z} \in \Gamma^{\boldsymbol{T}}.$$
Therefore
$$\begin{aligned}\hat{\mu}(\boldsymbol{z}) &= \sum_{\boldsymbol{n}\in\mathcal{P}(\boldsymbol{T})} \boldsymbol{z}^{-\boldsymbol{n}}\mu(\overline{V}_{n,\,T})\\ &= \sum_{\boldsymbol{n}\in\mathcal{P}(\boldsymbol{T})} \boldsymbol{z}^{-\boldsymbol{n}}\mu_{\boldsymbol{T}}(\boldsymbol{n})\\ &= |\det \boldsymbol{T}|\hat{\mu}_{\boldsymbol{T}}(\boldsymbol{z}), \qquad \text{if } \boldsymbol{z} \in \Gamma^{\boldsymbol{T}}.\end{aligned}$$
Applying the Fourier inversion theorem to the last expression
$$\begin{aligned}\mu(\overline{V}_{n,\,T}) = \mu_{\boldsymbol{T}}(\boldsymbol{n}) &= \sum_{\boldsymbol{z}\in\Gamma^{\boldsymbol{T}}} \hat{\mu}_{\boldsymbol{T}}(\boldsymbol{z})\boldsymbol{z}^{\boldsymbol{n}}\\ &= \frac{1}{|\det \boldsymbol{T}|}\sum_{\boldsymbol{z}\in\Gamma^{\boldsymbol{T}}} \hat{\mu}(\boldsymbol{z})\boldsymbol{z}^{\boldsymbol{n}}.\end{aligned}$$
Taking limits in the net with respect to the preorder relation of $\mathcal{GL}(k, \mathbb{Z})$
$$\begin{aligned}&\lim_{\boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z})} \frac{1}{|\det \boldsymbol{T}|}\sum_{\boldsymbol{z}\in\Gamma^{\boldsymbol{T}}} \hat{\mu}(\boldsymbol{z})\boldsymbol{z}^{\boldsymbol{n}}\\ &\quad = \lim_{\boldsymbol{T}\in\mathcal{GL}(k,\,\mathbb{Z})} \mu(\overline{V}_{n,\,T}) = \mu(\{\boldsymbol{n}\}) = 0.\end{aligned}$$

On the other hand, if $\boldsymbol{T}$ is a diagonal matrix and its diagonal is $(t_1, \cdots, t_k) \in \mathbb{N}^k$
$$\begin{aligned}\frac{1}{|\det \boldsymbol{T}|}\sum_{\boldsymbol{z}\in\Gamma^{\boldsymbol{T}}} \hat{\mu}(\boldsymbol{z})\boldsymbol{z}^{\boldsymbol{n}} &= \frac{1}{t_1\cdots t_k}\sum_{m_1=0}^{t_1-1}\cdots\sum_{m_k=0}^{t_k-1}\\ &\quad\cdot f(e^{2\pi i(m_1/t_1)}, \cdots, e^{2\pi i(m_k/t_k)})\\ &\quad\cdot e^{2\pi i[(m_1n_1/t_1)+\cdots+(m_kn_k/t_k)]}\end{aligned}$$
is a Riemann sum in $\mathbb{T}^k$ of the integral
$$\begin{aligned}&\int_{[0,\,1]^k} f(e^{2\pi i s_1}, \cdots, e^{2\pi i s_k})e^{2\pi i \boldsymbol{s}\boldsymbol{n}}\,ds_1\cdots ds_k\\ &\quad = \int_{\mathbb{T}^k} f(\boldsymbol{w})\boldsymbol{w}^{\boldsymbol{n}}\,dw_1\cdots dw_k.\end{aligned}$$
Hence, the Fourier transform of f in $\mathbb{T}^k$ is zero
$$\hat{f}(\boldsymbol{n}) = \int_{\mathbb{T}^k} f(\boldsymbol{w})\boldsymbol{w}^{-\boldsymbol{n}}\,dw_1\cdots dw_k = 0, \qquad \text{for } \boldsymbol{n} \in \mathbb{Z}^k$$
which means that the function f is equal to zero almost everywhere in $\mathbb{T}^k$.

Proof of the Implication 3) ⇒ 1): If statement 3) holds, then ϕ is a filter and f is its transfer function. Let μ be the measure associated with ϕ, i.e., $\hat{\mu} = f$. The measure μ is the sum of two measures μ_1 and μ_2, where μ_1 is concentrated on $\mathbb{Z}^k$ and $\mu_2(\{\boldsymbol{n}\}) = 0$ for all $\boldsymbol{n} \in \mathbb{Z}^k$. Hence, the Fourier–Stieltjes transform of μ_1 may be extended to a continuous function in $\mathbb{T}^k$. Then, the function $\hat{\mu}_2 = \hat{\mu} - \hat{\mu}_1 = f - \hat{\mu}_1$ can be also extended to a continuous function in $\mathbb{T}^k$. It follows from the previous lemma that $\hat{\mu}_2$ is equal to zero. As a consequence, $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 = \hat{\mu}_1$; the uniqueness of the Fourier–Stieltjes transform implies that $\mu = \mu_1$. Therefore, μ is concentrated on $\mathbb{Z}^k$.

We want to emphasize the relation between this theorem and some well-known results in harmonic analysis. Since $\mathbb{T}^k$ is the dual group of $\mathbb{Z}^k$, the Fourier transform of an element of $\mathcal{L}^1(\mathbb{Z}^k)$ belongs to $\mathcal{C}(\mathbb{T}^k)$ (see [6, pp. 89–90, 93]).

Corollary: Let f be a complex function defined on Γ_k. The following statements are equivalent.

1) f is the transfer function of a filter whose associated measure is concentrated on $\mathbb{Z}^k$.
2) There exists an absolutely summable family $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ such that

$$f(z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}} z^{\boldsymbol{n}}.$$

A filter whose transfer function is

$$f(z) = \sum_{\boldsymbol{n}\in\mathbb{N}^k} a_{\boldsymbol{n}} z^{-\boldsymbol{n}} \qquad \text{for } z \in \Gamma_k$$

where $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^k}$ is a summable family of complex numbers, is called a *causal* or *analytic filter* (the function $z \in \Gamma_k \mapsto f(\overline{z})$ may be extended to an analytic function in the unit open polydisc and continuous in the closed polydisc). For this class of filters, the value of an output signal in a given time depends only on the values of the input signal until that precise moment. From the previous corollary, we have the following corollary.

Corollary: Let ϕ be a linear map from $sp(k)$ to $sp(k)$. The following statements are equivalent.

1) ϕ is an analytic filter.
2) ϕ is a filter associated with a measure concentrated on $\mathbb{N}^k$.

The structure of those filters whose transfer function is continuous on the k-dimensional torus is also useful in obtain some properties that become useful in the resolution of difference equations.

Proposition: Let ϕ be a filter whose associated measure is concentrated on $\mathbb{Z}^k$, and f is its transfer function. If $f(z) \neq 0$ for every $z \in \mathbb{T}^k$, then ϕ is an isomorphism, and ϕ^{-1} is also a filter whose associated measure is concentrated on $\mathbb{Z}^k$.

Proof: From Wiener's lemma (see [6, Sec. 4.5], [9, pp. 202–203, 226–229], and [15, pp. 266–267]), there exists a summable family $\{b_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ such that

$$\frac{1}{f(z)} = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} b_{\boldsymbol{n}} z^{\boldsymbol{n}}, \qquad z \in \mathbb{T}^k.$$

It is straightforward to prove that the filter ψ on $sp(k)$ with transfer function $1/f$ satisfies

$$\phi \circ \psi = \psi \circ \phi = \text{identity}.$$

Hence, ϕ is an isomorphism, and $\phi^{-1} = \psi$.

Difference equations are important not only for defining certain filters but also because they can serve as computational algorithms for realizing those systems.

Corollary: Let $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$, $\{b_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^k}$ be two absolutely summable families of complex numbers $q(z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} b_{\boldsymbol{n}} z^{-\boldsymbol{n}}$ and $p(z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^k} a_{\boldsymbol{n}} z^{-\boldsymbol{n}}$. The difference equation

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^k} b_{\boldsymbol{k}} y(\boldsymbol{n} - \boldsymbol{k}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^k} a_{\boldsymbol{j}} x(\boldsymbol{n} - \boldsymbol{j})$$

defines a filter Φ on $sp(k)$, $\underline{y} = \Phi\underline{x}$, if $q(z)$ has no zeros on $\mathbb{T}^k$. In this case, the measure associated with Φ is concentrated on $\mathbb{Z}^k$, and its transfer function is $f(z) = p(z)/q(z)$.

III. Conclusion

When we work with periodic signals (periodization of finite-extent signals, discrete Fourier transform), the set of all complex signals and even the space of bounded signals may be too large to handle easily. It is often advisable to restrict ourselves to the smaller space of signals that can be uniformly approximated by periodic signals, i.e., the space $sp(k)$ of semiperiodic signals.

We study filters on k-dimensional semiperiodic signals. These filters may be characterized in terms of Borel measures on a certain compact group Δ_k. Using the particular structure of the group, we prove that under certain regularity conditions on the Fourier–Stieltjes transform of the Borel measures on Δ_k, those measures are concentrated on $\mathbb{Z}^k$. In a more applied language, this will mean that filters whose transfer functions satisfies a rather weak regularity condition will be associated with a classical convolution filter. Finally, we prove that a difference equation can be solved on $sp(k)$ if $q(z) \neq 0$ on the k-dimensional torus. This improves on some known results (which demand q to be zero-free in $|z_j| \geq 1$) and will be useful in realizing digital filters with infinite-extent impulse responses.

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