

MULTIRESOLUTION ANALYSIS AND RADON MEASURES  
ON A LOCALLY COMPACT ABELIAN GROUP

FÉLIX GALINDO and JAVIER SANZ, Valladolid

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*Abstract.* A multiresolution analysis is defined in a class of locally compact abelian groups  $G$ . It is shown that the spaces of integrable functions  $\mathcal{L}^p(G)$  and the complex Radon measures  $M(G)$  admit a simple characterization in terms of this multiresolution analysis.

*Keywords:* multiresolution analysis, Radon measures, topological groups

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## 1. INTRODUCTION

One of the most fruitful ideas, both in theory and applications, in recent studies on harmonic analysis is the notion of multiresolution approximation. The concept is described by Y. Meyer [11]: “The idea of a multiresolution approximation enables us to combine analysis in the space variable with analysis in the frequency variable, while satisfying Heisenberg’s uncertainty principle. To be more precise, it is a question of approximating a general function  $f$  by a sequence of simple functions  $f_n$ .” So far multiresolution approximations have been mostly studied in the Euclidean  $n$ -dimensional space [10, 11], but the concept may be extended to other types of locally compact abelian (LCA) groups [7, 9] and the functions defined in them, which often describe important notions in physics and engineering; thus the potential applicability of this extension is considerable. We will show that the spaces of integrable functions  $\mathcal{L}^p(G)$  and complex Radon measures  $M(G)$  may be constructed in terms of a multiresolution analysis. Specifically, we will consider groups with a special structure:

We will assume that  $G$  is a locally compact abelian group containing a sequence  $(G_n)_{n=-\infty}^{\infty}$  of subgroups with the following properties:

- (i)  $G_n$  is open and compact;
- (ii)  $G_{n+1} \subset G_n$ ;
- (iii)  $\bigcup G_n = G$ ;
- (iv)  $\bigcap G_n = \{0\}$ .

As a matter of fact property iv) could be replaced by

- (iv')  $(G_n)_{n=-\infty}^{\infty}$  is a base of neighbourhoods at 0.

Since  $G_n$  is an open and compact subgroup, the quotient group  $G_{n+1}/G_n$  is finite and hence  $G/G_n$  is countable. We denote by  $G_{n,j}$  the cosets of  $G_n$ . Let  $H = \hat{G}$  be the dual group of  $G$  and  $H_n = G_n^\perp$  the annihilator of  $G_n$  in  $H$ . Since  $G_n$  is open, the quotient group  $G/G_n$  is discrete and its dual group  $H_n$  is compact; in a similar way, the compactness of  $G_n$  implies that  $\hat{G}/H_n$  is a discrete group and it follows that  $H_n$  is open. It is easy to prove that the sequence  $(H_n)_{n=-\infty}^{\infty}$  is increasing ( $H_n \subset H_{n+1}$ ),  $\bigcap H_n = \{0\}$  and  $\bigcup H_n = H$ , which means that the dual group  $H$  has also a suitable family of compact open subgroups. For further details about these groups, see [3].

Going back to the idea of multiresolution approximation, in our case the functions  $f_n$  approaching  $f$  will have the two features of simplicity and regularity in the following sense: their simplicity comes from the fact that they are completely determined when sampled on a fundamental domain of  $G/G_n$ . The regularity of the functions  $f_n$  corresponds to the fact that their Fourier transforms  $\hat{f}_n$  are supported by the compacts  $H_n$ . We remark that the product of the Haar measures of  $G_n$  and  $H_n$  is constant, reflecting the idea of the uncertainty principle.

Two examples are specially important:

1. The group  $G$  defined by

$$G = \prod_{-\infty}^0 (\mathbb{Z}/a_j\mathbb{Z}) \times \bigoplus_1^{\infty} (\mathbb{Z}/a_j\mathbb{Z}),$$

the product of a compact product and a discrete direct sum, where  $(a_i)_{i=-\infty}^{\infty}$  is a sequence of integers with  $a_i \geq 2$ . The group operation is performed coordinatewise, and the topology is the product topology.

For  $n \in \mathbb{Z}$ , let  $G_n$  be the subgroup

$$G_n = \{(x_j)_{j \in \mathbb{Z}} \in G : x_j = 0 \text{ for } j > n\}.$$

The family  $\{G_n\}$  is an increasing sequence of open and compact subgroups covering  $G$  that is a base of neighbourhoods at 0.

When  $a_i = 2$  for all  $i$ , it is possible to identify  $G$  with  $[0, \infty)$  as a measure space using the map  $|\cdot| : G \rightarrow [0, \infty)$  given by

$$x = (x_j)_{j \in \mathbb{Z}} \mapsto |x| = \sum_{j \in \mathbb{Z}} x_j 2^j.$$

This induces the Haar measure on  $G$ . The subgroup  $G_0$  is the Cantor dyadic group; questions of a measure-theoretic character concerning Walsh series on  $[0, 1]$  are the same as the corresponding questions about Fourier series on  $G_0$ .

2. The group  $\Omega_{\mathbf{a}}$  of the  $\mathbf{a}$ -adic numbers.

Let  $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$  be a sequence of positive integers, where each  $a_n$  is greater than or equal to 2. Consider the Cartesian product  $\prod_n \{0, 1, \dots, a_n - 1\}$ . Let  $G = \Omega_{\mathbf{a}}$  be the set of all  $\mathbf{x} = (x_n)$  in this space such that  $x_n$  is 0 for all  $n > n_0$ , where  $n_0$  is an integer that depends upon  $\mathbf{x}$ .

For  $\mathbf{x} = (x_n)$ ,  $\mathbf{y} = (y_n)$  in  $\Omega_{\mathbf{a}}$ , let  $\mathbf{z} = (z_n)$  be defined as follows. Suppose that  $x_{m_0} \neq 0$  and  $x_n = 0$  for  $n > m_0$ , and  $y_{n_0} \neq 0$  and  $y_n = 0$  for  $n > n_0$ . Then let  $z_n = 0$  for  $n > p_0 = \min(m_0, n_0)$ . Write  $x_{p_0} + y_{p_0} = t_{p_0} a_{p_0} + z_{p_0}$ , where  $z_{p_0} \in \{0, 1, \dots, a_{p_0} - 1\}$  and  $t_{p_0}$  is an integer. Suppose that  $z_{p_0}, z_{p_0-1}, \dots, z_k$  and  $t_{p_0}, t_{p_0-1}, \dots, t_k$  have been defined. Then write  $x_{k-1} + y_{k-1} + t_k = t_{k-1} a_{k-1} + z_{k-1}$ , where  $z_{k-1} \in \{0, 1, \dots, a_{k-1} - 1\}$  and  $t_{k-1}$  is an integer. This defines by induction a sequence  $\mathbf{z} = (z_n)$  in  $\Omega_{\mathbf{a}}$ , and we set  $\mathbf{x} + \mathbf{y} = \mathbf{z}$ . With the operation  $+$  defined in this way, the set  $\Omega_{\mathbf{a}}$  is an Abelian group.

For  $n \in \mathbb{Z}$ , let  $\Lambda_n$  be the set defined by

$$\Lambda_n = \{(x_j)_{j \in \mathbb{Z}} \in G : x_j = 0 \text{ for all } j > n\}.$$

The sets  $\{\Lambda_k\}$  define a topology on  $\Omega_{\mathbf{a}}$  which makes it an LCA group having the desired properties. See [5] for the development of harmonic analysis on these groups. Applications of groups of this type can be found in [4], [12].

2. A MULTIREOLUTION ANALYSIS OF  $\mathcal{L}^2(G)$

In the following let  $m$  be a Haar measure on  $G$  and  $\mathcal{L}^p(G)$  the spaces obtained with this measure.

A complex function  $f$  defined on  $G$  is  $G_n$ -periodic if

$$f(x + y) = f(x) \quad \text{for all } x \in G, \quad y \in G_n.$$

We denote by  $\mathcal{P}_n$  the set of the  $G_n$ -periodic functions. A function  $f$  in  $\mathcal{P}_n$  is constant on the cosets  $G_{n,j}$ , and we can write

$$f = \sum_j a_j \chi_{G_{n,j}},$$

where  $\chi_{G_{n,j}}$  is the characteristic function of  $G_{n,j}$ . It is easy to see that  $(\mathcal{P}_n)_{n=-\infty}^{\infty}$  is an increasing sequence of linear subspaces of infinite dimension in the space  $\mathcal{C}_u(G)$  of uniformly continuous functions on  $G$ . If  $\mathcal{P} = \bigcup_n \mathcal{P}_n$  is the set of  $G_n$ -periodic functions for some  $n$ , the Stone-Weierstrass theorem asserts that  $\mathcal{P} \cap \mathcal{C}_c(G)$  is dense in  $\mathcal{C}_0(G)$  with the supremum norm. Hence, the space  $\mathcal{P} \cap \mathcal{L}^p(G)$  is dense in  $\mathcal{L}^p(G)$  for  $1 \leq p < \infty$ .

Let us now consider the case  $p = 2$ . For  $n \in \mathbb{Z}$ , let  $V_n$  denote the linear subspace of  $G_n$ -periodic functions in  $\mathcal{L}^2(G)$ , i.e.,

$$V_n = \mathcal{P}_n \cap \mathcal{L}^2(G).$$

The family  $\{V_n\}_{n=-\infty}^{\infty}$  is a *multiresolution analysis (MRA)* of  $\mathcal{L}^2(G)$ ; this means that it satisfies the following properties:

- (MR1)  $V_n$  is a closed linear subspace of  $\mathcal{L}^2(G)$ ;
- (MR2)  $V_n \subset V_{n+1}$ ;
- (MR3)  $\bigcup_n V_n$  is dense in  $\mathcal{L}^2(G)$ ;
- (MR4)  $\bigcap_n V_n = \{0\}$ .

Some remarks are called for:

1. The family of functions  $\{\varphi_{n,j}\}_j$ , where

$$\varphi_{n,j} = \frac{1}{m(G_n)} \chi_{G_{n,j}},$$

is an orthogonal basis of  $V_n$ .

2. If we denote by  $\varphi_n$  the function

$$\varphi_n = \frac{1}{m(G_n)} \chi_{G_n},$$

the family  $\{\varphi_{n,j}\}_j$  is the set of all functions obtained by translating  $\varphi_n$ . Hence the subspaces  $V_n$  are translation invariant.

3. For the locally compact Cantor group, the Haar wavelet is constructed from this MRA ([7]). In general, we can expect to obtain a construction of wavelets only on groups where a dilation operator has been defined (as, for instance, in [7] and [9]).

4. Non-existence of a dilation operator has another consequence: we have not a unique scaling function, but scaling functions  $\varphi_n$  depending on the level. Certain “cascade” conditions give the change of scale in our case.

5. Finally we note that  $V_n$  can also be defined in the following way:

$$V_n = \{f \in \mathcal{L}^2(G) : \text{supp } \hat{f} \subset H_n\}.$$

In a certain sense, our multiresolution analysis is the only one possible: if we did not have a countable subgroup playing the role of the integer numbers in the real line (in the locally compact Cantor group,  $D = \{(x_j)_{j \in \mathbb{Z}} \in G : x_j = 0 \text{ for } j \leq 0\}$  plays this role), we ought to work with closed translation invariant linear subspaces  $V$ , i.e. subspaces given by  $\hat{V} = \chi_E \mathcal{L}^2(H)$ , where  $E$  is a Borel measurable subset of  $H$  [8].

If  $f \in \mathcal{L}^2(G)$  and we define

$$\alpha_f(n, j) = \langle f, \varphi_{n,j} \rangle = \frac{1}{m(G_n)} \int_{G_{n,j}} f(x) dm(x),$$

the function

$$S_n f = \sum_j \alpha_f(n, j) \chi_{G_{n,j}} = m(G_n) \sum_j \alpha_f(n, j) \varphi_{n,j}$$

is the orthogonal projection of  $f$  onto  $V_n$ . Note that if  $f$  is  $G_n$ -periodic, the coefficient  $\alpha_f(n, j)$  is the value that  $f$  takes in  $G_{n,j}$ : indeed, for  $x \in G_{n,j}$ ,

$$\alpha_f(n, j) = \frac{1}{m(G_n)} \int_{G_{n,j}} f(y) dm(y) = \frac{1}{m(G_n)} f(x) m(G_{n,j}) = f(x).$$

The numbers  $\alpha_f(n, j)$  so defined satisfy the following cascade conditions: if  $r \geq s$ , then

$$\alpha_f(s, k) = \frac{m(G_r)}{m(G_s)} \sum_{j: G_{r,j} \subset G_{s,k}} \alpha_f(r, j).$$

The properties of the functions  $\varphi_n$  enable us to extend the multiresolution approximation  $V_n$  of  $\mathcal{L}^2(G)$  to other functions spaces. We note that  $\alpha_f(n, j)$  is perfectly defined if  $f \in \mathcal{L}^p(G)$ ,  $1 \leq p \leq \infty$  (in fact, it would be enough if  $f$  were a locally integrable function).

Two questions arise in a natural way. First, is it possible to characterize the spaces  $\mathcal{L}^p(G)$  by controlling the size of the coefficients  $\alpha_f(n, j)$ ? Second, does the corresponding *moment problem* have a solution, i.e., if  $\alpha = \{\alpha(n, j)\}$  is a family of complex numbers, can we find  $f \in \mathcal{L}^p(G)$  such that

$$\langle f, \varphi_{n,j} \rangle = \alpha(n, j)?$$

### 3. THE SPACES $\mathcal{L}^p(G)$

Let  $f$  be a function of  $\mathcal{L}^p(G)$  with  $1 \leq p \leq \infty$ . We define

$$\alpha_f(n, j) = \langle f, \varphi_{n,j} \rangle = \frac{1}{m(G_n)} \int_{G_{n,j}} f(x) \, dm(x).$$

The first point to notice is that for a  $G_n$ -periodic function  $f$ , the norm  $\|f\|_p$  is easy to calculate in terms of  $\alpha_f(n, j)$ .

**Lema 3.1.** *Let  $p$  be a real number,  $1 \leq p \leq \infty$ . If  $f \in \mathcal{P}_n \cap \mathcal{L}^p(G)$ , then*

$$(1) \quad \|f\|_p = \begin{cases} \left( m(G_n) \sum_j |\alpha_f(n, j)|^p \right)^{1/p}, & \text{if } 1 \leq p < \infty; \\ \sup_j |\alpha_f(n, j)|, & \text{if } p = \infty. \end{cases}$$

This fact will allow us to identify  $\mathcal{L}^p(G)$  with a space of families  $\{\alpha(n, j)\}$  of complex numbers. The last remark in the previous section and Lemma 3.1 take us to the following definition:

**Definition 3.2.** An *infinite tree* is a family  $\alpha = \{\alpha(n, j)\}$  of complex numbers satisfying the cascade conditions

$$(2) \quad \alpha(s, k) = \frac{m(G_r)}{m(G_s)} \sum_{j: G_{r,j} \subset G_{s,k}} \alpha(r, j) \text{ for } r \geq s.$$

The set of infinite trees with pointwise operations is a linear space.

**Lema 3.3.** *Let  $p$  be a real number,  $1 \leq p < \infty$ , and  $f \in \mathcal{L}^p(G)$ . Then  $\alpha_f = \{\alpha_f(n, j)\}$  is an infinite tree and*

$$(3) \quad \|f\|_p = \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_f(n, j)|^p \right)^{1/p}.$$

**P r o o f.** If  $r \geq s$ , the equation

$$\chi_{G_{s,k}} = \sum_{j: G_{r,j} \subset G_{s,k}} \chi_{G_{r,j}}$$

gives, after some easy calculations, the cascade conditions (2).

To show (3), we define

$$(4) \quad S_n f = \sum_j \alpha_f(n, j) \chi_{G_{n,j}}.$$

We note that  $S_n f = f * \varphi_n$ . Since  $\{\varphi_n\}_{n=-\infty}^{\infty}$  is an approximation of unity, we have

$$(5) \quad \|S_n f\|_p \leq \|f\|_p,$$

$$(6) \quad \lim_{n \rightarrow \infty} \|S_n f - f\|_p = 0.$$

By Lemma 3.1,

$$\|f\|_p = \sup_{n \in \mathbb{Z}} \|S_n f\|_p = \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_f(n, j)|^p \right)^{1/p}.$$

□

Surprisingly, for  $1 < p < \infty$  these conditions are also sufficient to characterize  $\mathcal{L}^p(G)$  and we do not need any “convergence” condition that, however, is necessary for  $p = 1$ . We state this fact as follows:

**Theorem 3.4.** *Let  $p$  be a real number,  $1 < p < \infty$ . If  $f \in \mathcal{L}^p(G)$ , then the family  $\alpha_f = \{\alpha_f(n, j)\}$  is an infinite tree and*

$$(7) \quad \|f\|_p = \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_f(n, j)|^p \right)^{1/p}.$$

*Conversely, if  $\alpha = \{\alpha(n, j)\}$  is an infinite tree and*

$$(8) \quad \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha(n, j)|^p \right)^{1/p} < \infty,$$

*there exists a unique  $f \in \mathcal{L}^p(G)$  such that*

$$(9) \quad \alpha(n, j) = \langle f, \varphi_{n,j} \rangle.$$

**Proof.** We have proved the first part of the statement in Lemma 3.3. Now we will prove the second part. Assume that  $\alpha = \{\alpha(n, j)\}$  is an infinite tree that satisfies (8). We consider the sequence of functions  $(f_n)_{n=-\infty}^{\infty}$  defined by

$$(10) \quad f_n = \sum_j \alpha(n, j) \chi_{G_{n,j}}.$$

By Lemma 3.1 and (8), the sequence  $(f_n)_{n=-\infty}^{\infty}$  is bounded in  $\mathcal{L}^p(G)$ . Applying the Alaoglu theorem ([2]), there exists a subsequence  $(f_{n_l})_{l=-\infty}^{\infty}$  weakly convergent to a function  $f$  in  $\mathcal{L}^p(G)$ . Then,

$$\alpha_f(n, k) = \langle f, \varphi_{n,k} \rangle = \lim_{l \rightarrow \infty} \langle f_{n_l}, \varphi_{n,k} \rangle = \lim_{l \rightarrow \infty} \frac{m(G_{n_l})}{m(G_n)} \sum_{j: G_{n_l, j} \subset G_{n,k}} \alpha(n_l, j) = \alpha(n, k),$$

which shows (9). The uniqueness is a consequence of (7): indeed, let  $g$  be another function of  $\mathcal{L}^p(G)$  such that  $\alpha(n, j) = \langle g, \varphi_{n,j} \rangle$ . Then  $\alpha_{f-g}(n, j) = 0$  for all  $n, j$ . Applying (7), we have  $\|f - g\|_p = 0$ .  $\square$

The last proof does not work in  $\mathcal{L}^1(G)$ , since this space is not a dual one. An example shows that condition (8) is not sufficient for  $p = 1$ : indeed, the family  $\alpha = \{\alpha(n, j)\}$  given by

$$\alpha(n, j) = \begin{cases} 1/m(G_n) & \text{if } j = 0, \\ 0 & \text{if } j \neq 0, \end{cases}$$

is an infinite tree that satisfies (8) for  $p = 1$ , and the sequence defined by (10) is the sequence of the scaling functions  $(\varphi_n)_{n=-\infty}^{\infty}$  that converges weakly to the Dirac measure  $\delta_0$ . We must add a convergence condition.

**Theorem 3.5.** *If  $f \in \mathcal{L}^1(G)$ , then the family  $\alpha_f = \{\alpha_f(n, j)\}$  is an infinite tree that satisfies the following two properties:*

$$(11) \quad \|f\|_1 = \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_f(n, j)| \right),$$

and for every  $\varepsilon > 0$  there exists  $N \in \mathbb{Z}$  such that

$$(12) \quad \left( m(G_r) \sum_k \sum_{j: G_{r,j} \subset G_{s,k}} |\alpha_f(r, j) - \alpha_f(s, k)| \right) < \varepsilon, \quad r > s \geq N.$$

Conversely, if  $\alpha = \{\alpha(n, j)\}$  is an infinite tree such that

$$(13) \quad \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha(n, j)| \right) < \infty,$$

and for every  $\varepsilon > 0$  there exists  $N \in \mathbb{Z}$  such that

$$(14) \quad \left( m(G_r) \sum_k \sum_{j: G_{r,j} \subset G_{s,k}} |\alpha(r, j) - \alpha(s, k)| \right) < \varepsilon, \quad r > s \geq N,$$



then there exists a unique  $f \in \mathcal{L}^1(G)$  such that

$$(15) \quad \alpha(n, j) = \langle f, \varphi_{n, j} \rangle.$$

*Proof.* It is enough to note that (14) is equivalent to the Cauchy condition in  $\mathcal{L}^1(G)$  for the sequence  $(f_n)_{n=-\infty}^{\infty}$  defined by (10); now we repeat the proof of Theorem 3.4.  $\square$

What happens in the case  $p = \infty$ ? The key to this question is the fact that, in general, the sequence  $(S_n f)_{n=-\infty}^{\infty}$  defined by (4) need not converge in  $\mathcal{L}^\infty(G)$ , because its limit should be a bounded uniformly continuous function on  $G$ .

**Theorem 3.6.** *If  $f \in \mathcal{L}^\infty(G)$ , then the family  $\alpha_f = \{\alpha_f(n, j)\}$  is an infinite tree and*

$$(16) \quad \|f\|_\infty = \sup_{n \in \mathbb{Z}} \left( \sup_j |\alpha_f(n, j)| \right).$$

*Conversely, if  $\alpha = \{\alpha(n, j)\}$  is an infinite tree and*

$$(17) \quad \sup_{n \in \mathbb{Z}} \left( \sup_j |\alpha(n, j)| \right) < \infty,$$

*then there exists a unique  $f \in \mathcal{L}^\infty(G)$  such that*

$$(18) \quad \alpha(n, j) = \langle f, \varphi_{n, j} \rangle.$$

*Proof.* If  $f \in \mathcal{L}^\infty(G)$ , as in the proof of Lemma 3.3, then

$$\sup_j |\alpha(n, j)| = \|S_n f\|_\infty \leq \|f\|_\infty.$$

Now, let  $\alpha = \{\alpha(n, j)\}$  be an infinite tree such that

$$\sup_{n \in \mathbb{Z}} \left( \sup_j |\alpha(n, j)| \right) < \infty.$$

If  $(f_n)_{n=-\infty}^{\infty}$  is the sequence of functions in  $\mathcal{L}^\infty(G)$  defined by (10) in the proof of Theorem 3.4, we note the following: for every  $g$  in  $\mathcal{L}^1$ , the sequence

$$\left( \langle g, f_n \rangle \right)_{n=-\infty}^{\infty} = \left( m(G_n) \sum_j \alpha(n, j) \alpha_g(n, j) \right)_{n=-\infty}^{\infty}$$

is a Cauchy sequence of complex numbers. Indeed, if  $r, s \in \mathbb{Z}, r \geq s$ , we have

$$\begin{aligned}
 & \left| m(G_r) \sum_j \alpha(r, j) \alpha_g(r, j) - m(G_s) \sum_k \alpha(s, k) \alpha_g(s, k) \right| \\
 &= \left| m(G_r) \sum_{j \in \mathbb{N}_r} \alpha(r, j) \alpha_g(r, j) - m(G_s) \sum_k \left( \frac{m(G_r)}{m(G_s)} \sum_{j: G_{r,j} \subset G_{s,k}} \alpha(r, j) \right) \alpha_g(s, k) \right| \\
 &= \left| m(G_r) \sum_j \alpha(r, j) \alpha_g(r, j) - m(G_r) \sum_k \left( \sum_{j: G_{r,j} \subset G_{s,k}} \alpha(r, j) \right) \alpha_g(s, k) \right| \\
 &\leq m(G_r) \sum_k \sum_{j: G_{r,j} \subset G_{s,k}} |\alpha(r, j)| |\alpha_g(r, j) - \alpha_g(s, k)| \\
 &\leq \left( m(G_r) \sum_k \sum_{j: G_{r,j} \subset G_{s,k}} |\alpha_g(s, k) - \alpha_g(r, j)| \right) \sup_j |\alpha(r, j)|,
 \end{aligned}$$

and we can apply Theorem 3.5 and (17). In particular, if  $g$  is  $G_n$ -periodic, the limit is

$$m(G_n) \sum_j \alpha(n, j) \alpha_g(n, j).$$

Hence, we can define a map  $\varphi$  from  $\mathcal{L}^1(G)$  to  $\mathbb{C}$  by

$$\varphi(g) = \lim_{n \rightarrow \infty} m(G_n) \sum_j \alpha(n, j) \alpha_g(n, j).$$

Since  $\varphi$  is an element of the dual space of  $\mathcal{L}^1(G)$ , there exists a unique function  $f \in \mathcal{L}^\infty(G)$  such that

$$\varphi(g) = \int_G g f \, dm, \quad g \in \mathcal{L}^1(G).$$

If we consider the function  $g = \chi_{G_{n,j}}$ , then

$$\int_{G_{n,j}} f \, dm = \varphi(\chi_{G_{n,j}}) = m(G_n) \alpha(n, j).$$

Moreover,

$$\|f\|_\infty = \|\varphi\| \leq \sup_{n \in \mathbb{Z}} \|f_n\|_\infty = \sup_{n \in \mathbb{Z}} \left( \sup_j |\alpha(n, j)| \right).$$

□

This proof could also be made by a compactness argument, but we think it worthwhile to obtain this result using only that  $\mathcal{L}^\infty(G)$  is the dual space of  $\mathcal{L}^1(G)$  and Theorem 3.5.

**Remark.** If  $\mathcal{B}_n$  is the smallest  $\sigma$ -algebra containing  $\{G_{n,j}\}_j$ , it is interesting to note that the function  $S_n f$  defined by (4) is the conditional expectation of  $f$  given  $\mathcal{B}_n$ . On the other hand, if  $\alpha$  is an infinite tree, the family of functions  $(f_n^\alpha)_{n=-\infty}^\infty$  defined by (10) is a martingale relative to  $\mathcal{B}_n$  (see, for instance, [1]).

#### 4. RADON MEASURES ON $G$

In the previous section, we have seen that, in general, infinite trees  $\alpha$  with

$$\sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha(n, j)| \right) < \infty$$

do not correspond to functions of  $\mathcal{L}^1(G)$ . It is natural to ask if some special space is obtained when we consider infinite trees under this sole condition. We will see that this is the space  $M(G)$  of the complex Radon measures defined on  $G$ .

Let  $\mu$  be an element of  $M(G)$ . We define

$$(19) \quad \alpha_\mu(n, j) = \langle \mu, \varphi_{n,j} \rangle = \int_G \varphi_{n,j} \, d\mu.$$

**Lema 4.1.** *Let  $\mu$  be an element of  $M(G)$ . Then the family  $\alpha_\mu = \{\alpha_\mu(n, j)\}$  is an infinite tree and*

$$(20) \quad \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_\mu(n, j)| \right) \leq \|\mu\|.$$

*Proof.* If  $\mu \in M(G)$ , it is straightforward to show that  $\alpha_\mu$  is an infinite tree, and, noting that

$$\mu(G_{n,j}) = m(G_n) \alpha_\mu(n, j),$$

we have from the definition of total variation of  $\mu$  that

$$m(G_n) \sum_j |\alpha_\mu(n, j)| \leq \|\mu\|, \quad n \in \mathbb{Z}.$$

□

**Theorem 4.2.** *Let  $\mu$  be an element of  $M(G)$ . Then the family  $\alpha_\mu = \{\alpha_\mu(n, j)\}$  is an infinite tree and*

$$(21) \quad \|\mu\| = \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha_\mu(n, j)| \right).$$

Conversely, if  $\alpha = \{\alpha(n, j)\}$  is an infinite tree and

$$(22) \quad \sup_{n \in \mathbb{Z}} \left( m(G_n) \sum_j |\alpha(n, j)| \right) < \infty,$$

there exists a unique  $\mu \in M(G)$  such that

$$(23) \quad \alpha(n, j) = \langle \mu, \varphi_{n,j} \rangle.$$

*Proof.* Since  $M(G)$  is the dual of the space  $C_0(G)$  of continuous functions on  $G$  which vanish at infinity, we can repeat the proof of Theorem 3.4.  $\square$

Finally we remark that certain properties of a measure  $\mu$  of  $M(G)$  are reflected in its infinite tree  $\alpha_\mu = \{\alpha_\mu(n, j)\}$ . For instance, Theorem 3.5 characterizes the absolutely continuous measures with respect to the Haar measure  $m$ . We give some others examples:

**Proposition 4.3.** *Let  $\mu \in M(G)$ . Then the limit*

$$h(x) = \lim_{n \rightarrow \infty} \alpha_\mu(n, j(x)),$$

*exists almost everywhere (where  $j(x)$  is the unique index such that  $x \in G_{n,j(x)}$ ). Moreover,  $h$  is the Radon-Nikodym derivative  $d\mu/dm$  of  $\mu$  with respect to the Haar measure  $m$ .*

*Proof.* This is a consequence of Possel's theorem ([13], pp. 215–216) applied to the net  $\mathcal{H} = \bigcup_{n=-\infty}^{\infty} \mathcal{H}_n$ , where  $\mathcal{H}_n$  is the family of cosets of  $G_n$ .  $\square$

**Corollary 4.4.** *Let  $\mu \in M(G)$ . Then  $\mu$  is singular to  $m$  if, and only if,*

$$\lim_{n \rightarrow \infty} \alpha_\mu(n, j(x)) = 0 \text{ almost everywhere.}$$

**Proposition 4.5.** *Let  $\mu \in M(G)$ . Then  $\mu$  is diffuse (i.e.  $\mu(\{x\}) = 0$  for all  $x \in G$ ) if and only if*

$$\lim_{n \rightarrow \infty} m(G_n) \alpha_\mu(n, j(x)) = 0$$

*uniformly on every compact of  $G$ .*

*Proof.* It is enough to prove the proposition for nonnegative measures, as the Jordan components of a diffuse measure are also diffuse. Consider functions

$$g_n(x) = m(G_n) \sum_j \alpha_\mu(n, j) \chi_{G_{n,j}}, \quad x \in G.$$

Then  $(g_n)_{n=-\infty}^{\infty}$  is a decreasing sequence of continuous functions. Since  $\mu$  is diffuse,  $(g_n(x))_{n=-\infty}^{\infty}$  converges to zero for every  $x \in G$ :

$$g_n(x) = \mu(G_{n,j(x)}) \rightarrow \mu(\{x\}) = 0 \quad \text{as } n \rightarrow \infty.$$

Now, by Dini's theorem, the convergence is uniform on every compact of  $G$ . □

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*Authors' address:* Departamento de Análisis Matemático (Facultad de Ciencias), Universidad de Valladolid, 47005 Valladolid, Spain, e-mail: fgalindo@am.uva.es, jsanzg@am.uva.es.