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Uniform growth rates for the magnetic field in a kinematic dynamo

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Abstract. The growth rate of a magnetic field within a conducting fluid with a certain fixed kinematic velocity is an indication of the efficiency of the dynamo effect. A possible way to measure this growth rate consists of considering the maximum size of the magnetic field for all possible initial conditions and for every instant of time. Both the maximum and the minimum in time of these values are completely characterized here in terms of the numerical range and the spectrum of the induction operator for all the usual norms. Among the consequences of these results it is found that the classical bounds obtained by energy inequalities are optimal, that the maximal uniform growth rate for the energy norm essentially coincides for the magnetic field and the perturbed velocity, and that the minimal uniform growth rate for the magnetic field is precisely the classical maximal growth rate.

1. Introduction

If we have a conducting fluid with a steady velocity $\boldsymbol{u}$ and consider the evolution of a magnetic field $\boldsymbol{B}(t, \boldsymbol{x})$ which at $t = 0$ coincides with $\boldsymbol{B}_0$, we must first study the induction equation

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= \eta \Delta \boldsymbol{B} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\ \boldsymbol{B}(0) &= \boldsymbol{B}_0 \end{aligned} \tag{1}$$

plus the appropriate boundary conditions; η stands for the resistivity. If we take $\boldsymbol{u}$ as a fixed field, this is a linear parabolic differential system and as such much more tractable than the whole of the magnetohydrodynamical equations. Of course, the effect of the magnetic field upon the velocity through the Lorentz force cannot be ignored for long and the correctness of the evolution described by (1) ends when this kinematic dynamo saturates and the linear description fails. However, kinematic dynamo theory is far from simple because of the usually very complicated velocity field. To study the possible growth of the magnetic field (which, by general theorems on C_0 semigroups, is at most exponential; see, e.g., [1]), it is classically defined the *maximal growth rate*:

$$\sigma_{\eta,\boldsymbol{u}} = \sup_{\boldsymbol{B}_0} \left(\lim_{t\to\infty} \frac{\log \|\boldsymbol{B}(t)\|}{t} \right). \tag{2}$$

$\boldsymbol{u}$ is said to originate a *fast dynamo* if $\inf_{\eta>0} \sigma_{\eta,\boldsymbol{u}} > 0$. (See [2] for a complete study.) In principle, the value of $\sigma_{\eta,\boldsymbol{u}}$ depends upon the norm we are considering for $\boldsymbol{B}$, but since the dynamo operator A defined by the right-hand side of (1), for all the usual norms [3], has a discrete spectrum accumulating at $-\infty$, the value of $\sigma_{\eta,\boldsymbol{u}}$ can easily be shown to be the real part of the eigenvalue of A with largest real part. This is a purely algebraic amount and therefore independent of the norm.

However, the above definition is not altogether satisfactory. If we know that the kinematic growth will stop at some saturation time, why the limit when $t \to \infty$? Because for finite time the value will depend heavily on the initial condition $\boldsymbol{B}_0$, not on the real dynamo process. This can be avoided by using the *uniform growth rate*

$$F(t) = \sup_{\|\boldsymbol{B}_0\|=1} \frac{\log \|\boldsymbol{B}(t)\|}{t}. \tag{3}$$

The bound on the initial condition is intended to avoid spurious size due to a simple scaling of $\boldsymbol{B}_0$. Now we define the *maximal uniform growth rate* as

$$\Gamma_{\eta,\boldsymbol{u}} = \sup_{t>0} F(t) \tag{4}$$

and the *minimal uniform growth rate*

$$\gamma_{\eta,\boldsymbol{u}} = \inf_{t>0} F(t). \tag{5}$$

From now on we will omit the parameters η and $\boldsymbol{u}$ if there is no danger of confusion. The fact that these values are generally different from the classical maximal growth rate and provide a different insight into the dynamo process may be understood, without any calculation, by the following argument. Imagine, as is often the case, that every single solution of the induction equation tends ultimately to zero, then $\sigma_{\eta,\boldsymbol{u}} \leqslant 0$. However, it is possible that considerable transient growth may occur before the ultimate decline. If different initial conditions have these transient growth periods at different intervals of time, the maximal uniform growth rate does not need to be negative. Suppose that we want to study the magnetic field magnitude for a given flow and (as indeed is the most usual realistic situation) we have no control upon the initial condition, except that it is bounded by a certain amount, we could expect that by having $\sigma_{\eta,\boldsymbol{u}} < 0$ that the field would vanish, and yet to observe it for a 1000 years without any indication of slackening. If $\Gamma_{\eta,\boldsymbol{u}} < 0$, however, the field will certainly decrease exponentially.

It is our purpose to find Γ and γ in terms of known values associated with the operator A. To this end, we briefly review the notion of the *numerical range* of an operator (see, e.g., [4,5]). We assume A is a closed operator defined in a dense domain $D(A)$ of the Banach space X, and let X^* be its dual space, formed by the continuous linear functionals defined on X. For every $x \in X$, we define

$$C(x) = \{v \in X^*/\|v\| = \|x\|, \langle v, x\rangle = \|x\|^2\}. \tag{6}$$

This set is nonempty by the Hahn–Banach theorem. In many cases it is formed by a single element: for instance, if X is a Hilbert space, identified with its dual by the internal product, $C(x) = \{x\}$. For the space $L^p(\Omega)$ of p-integrable functions on a domain Ω, $1 < p < \infty$, whose dual is $L^q(\Omega)$ (with $1/p + 1/q = 1$),

$$C(f) = \{\bar{f}|f|^{p-2}\|f\|^{2-p}\} \tag{7}$$

and for the space of continuous functions on a compact set $\mathcal{C}(K)$, let M be the set of positive measures supported on $\{z \in K/|f(z)| = \|f\|_\infty\}$. Then $C(f)$ is formed by the Borel measures λ such that $\mathrm{d}\lambda = \bar{f}\,\mathrm{d}\mu$ with $\mu \in M$. The action of such a measure on g is

$$\langle \lambda, g\rangle = \int_K g\,\mathrm{d}\lambda = \int_K g\bar{f}\,\mathrm{d}\mu. \tag{8}$$

The *numerical range* of A is defined by

$$W(A) = \{\langle v, Ax\rangle : x \in D(A), \|x\| = 1, v \in C(x)\}. \tag{9}$$

In particular, for a Hilbert space,

$$W(A) = \{(Ax, x) : x \in D(A), \|x\| = 1\}. \tag{10}$$

A lot is known about the numerical range and its relation to the spectrum of A (see the references above) particularly for bounded operators (which is not our case) in Hilbert spaces (which is not always our case). For instance, the spectrum $\sigma(A)$ is always contained in the closure of $W(A)$.

Finally, let us recall some properties of parabolic operators such as A, (see, e.g., [6]). The family of operators $T(t) : t \geqslant 0$ which take $\boldsymbol{B}_0$ to $\boldsymbol{B}(t)$ form a *semigroup* in $\mathcal{L}(X)$ by the obvious properties $T(0) = I$, $T(t+s) = T(t)T(s)$. In fact, $T(t)$ is a compact operator for $t > 0$, which essentially means that the mapping $t \to T(t)$ is continuous from $[0, \infty)$ to $\mathcal{L}(X)$ and that the resolvent operator $(\lambda - A)^{-1} = (\lambda I - A)^{-1}$ is compact for $\lambda \notin \sigma(A)$ (see, e.g., [7]).

In addition, $\|(\lambda - A)^{-1}\| \to 0$ when $\mathrm{Re}\,\lambda \to \infty$. In our case, A is the perturbation of a self-adjoint operator, which also implies $\|(\lambda - A)^{-1}\| \to 0$ when $|\mathrm{Im}\,\lambda| \to \infty$. The mapping $x \to T(t)x$ is differentiable at $t = 0$ only when $x \in D(A)$. Its differential is Ax. Finally, $T(t)$ may be expressed by an inverse-Laplace transform

$$T(t)\boldsymbol{B}_0 = \int_{r-\mathrm{i}\infty}^{r+\mathrm{i}\infty} \mathrm{e}^{t\lambda}(\lambda - A)^{-1}\boldsymbol{B}_0 \,\mathrm{d}\lambda. \tag{11}$$

Finally, we must caution the reader that such an obviously important problem as the growth of solutions of linear evolution equations associated with non-normal operators could hardly have been left undisturbed. It has long being acknowledged that for operators such as A the resolvent $(z - A)^{-1}$ may be small in norm for points z far from the spectrum, and for such z there is a kind of pseudoresonance (see [8] and references therein). The theorem in section 3 on the minimal growth rate does not hold for all operators [9], although it is true if A has a discrete spectrum [1], such as it is our case; and the theorem in section 2 on the maximal growth may be proved with more generality by functional analytic arguments [7]. However, a separate treatment for the kinematic dynamo is, in our opinion, justified because by the simplicity of the arguments one does not need to refer to abstract mathematical theorems, and the applications described in section 4 are unique to this problem. A related study with a somewhat different language is presented in [10].

2. The maximal uniform growth rate

Since

$$F(t) = \sup_{\|\boldsymbol{B}_0\|=1} \frac{\log\|\boldsymbol{B}(t)\|}{t} = \frac{\log\|T(t)\|}{t}$$

and $\|T(t+s)\| \leqslant \|T(t)\|\|T(s)\|$, the function in the numerator is a continuous subadditive one: $g(t+s) \leqslant g(t) + g(s)$. In this case,

$$\begin{aligned} \sup_{t>0} \frac{g(t)}{t} &= \lim_{t\to 0} \frac{g(t)}{t} \\ \inf_{t>0} \frac{g(t)}{t} &= \lim_{t\to\infty} \frac{g(t)}{t}. \end{aligned} \tag{12}$$

(see, e.g., [11]). Now, since $\|T(0)\| = 1$,

$$\lim_{t\to 0} \frac{\log\|T(t)\|}{t} = \lim_{t\to 0} \frac{\|T(t)\| - 1}{t} = \lim_{t\to 0} \sup_{\|\boldsymbol{B}\|=1} \frac{\|T(t)\boldsymbol{B}\| - 1}{t} \tag{13}$$

and since $D(A)$ is dense in X, $D(A) \cap \{\|\boldsymbol{B}\| = 1\}$ is dense in the unit sphere, so

$$\Gamma = \lim_{t\to 0} \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1)} \frac{\|T(t)\boldsymbol{B}\| - 1}{t}. \tag{14}$$

Now, by the theorem of Hahn–Banach,

$$\Gamma = \lim_{t\to 0} \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1)} \sup_{(v\in X^*, \|v\|=1)} \frac{\mathrm{Re}\, \langle v, T(t)\boldsymbol{B}\rangle - 1}{t}. \tag{15}$$

As stated before, $t \to T(t)\boldsymbol{B}$ is differentiable at $t = 0$; hence

$$T(t)\boldsymbol{B} = \boldsymbol{B} + tA\boldsymbol{B} + tU_t(\boldsymbol{B})$$

where $\|U_t(\boldsymbol{B})\| \to 0$ when $t \to 0$. Thus

$$\frac{\mathrm{Re}\, \langle v, T(t)\boldsymbol{B}\rangle - 1}{t} = \frac{\mathrm{Re}\, \langle v, \boldsymbol{B}\rangle - 1}{t} + \mathrm{Re}\, \langle v, A\boldsymbol{B}\rangle + \mathrm{Re}\, \langle v, U_t\boldsymbol{B}\rangle. \tag{16}$$

Since $\|\boldsymbol{B}\| = \|v\| = 1$, $\mathrm{Re}\, \langle v, \boldsymbol{B}\rangle \leqslant 1$ always. If it is strictly smaller than 1, the limit is $-\infty$, so when taking the supremum we may restrict ourselves to those $v \in X^*$ with $\mathrm{Re}\, \langle v, \boldsymbol{B}\rangle = 1$. Since $|\langle v, \boldsymbol{B}\rangle| \leqslant 1$, necessarily $\langle v, \boldsymbol{B}\rangle = 1$ and $\mathrm{Re}\, \langle v, \boldsymbol{B}\rangle - 1 = 0$; hence $v \in C(\boldsymbol{B})$. We are left with

$$\Gamma = \lim_{t\to 0} \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1)} \sup_{(v\in C(\boldsymbol{B}))} (\mathrm{Re}\, \langle v, A\boldsymbol{B}\rangle + \mathrm{Re}\, \langle v, U_t\boldsymbol{B}\rangle). \tag{17}$$

Now, $\langle v, U_t\boldsymbol{B}\rangle$ tends to zero for every $\boldsymbol{B} \in D(A)$; however, it does not tend uniformly to zero for all $\boldsymbol{B} \in D(A)$ with $\|\boldsymbol{B}\| = 1$, so we cannot simply cancel the last term. We know, however, that the limit is uniform when we impose a common bound in the *graph norm* $\|\boldsymbol{B}\|+\|A\boldsymbol{B}\|$ [12]; i.e.,

$$\begin{aligned} &\lim_{t\to 0} \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1, \|\boldsymbol{B}\|+\|A\boldsymbol{B}\|\leqslant M)} \sup_{(v\in C(\boldsymbol{B}))} (\mathrm{Re}\, \langle v, A\boldsymbol{B}\rangle + \mathrm{Re}\, \langle v, U_t\boldsymbol{B}\rangle) \\ &\qquad = \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1, \|\boldsymbol{B}\|+\|A\boldsymbol{B}\|\leqslant M)} \sup_{(v\in C(\boldsymbol{B}))} \mathrm{Re}\, \langle v, A\boldsymbol{B}\rangle. \end{aligned} \tag{18}$$

Unfortunately, the set of functions bounded in the graph norm by a certain constant is certainly not dense in the unit sphere of X: as a matter of fact it is compact in X [12]. We have reached the limits of a general argument, and to proceed further we need to turn to specific properties of the induction operator A.

In the inverse-Laplace integral expressing $T(t)$,

$$T(t)\boldsymbol{B} = \int_{r-\mathrm{i}\infty}^{r+\mathrm{i}\infty} \mathrm{e}^{t\lambda}(\lambda - A)^{-1}\boldsymbol{B}\, \mathrm{d}\lambda \tag{19}$$

we may deform the contour of integration by using Cauchy's theorem and the already-mentioned property $\|(\lambda - A)^{-1}\| \to 0$ when $|\mathrm{Im}\, \lambda| \to \infty$. We find

$$T(t)\boldsymbol{B} = \int_{-k-\mathrm{i}\infty}^{-k+\mathrm{i}\infty} \mathrm{e}^{t\lambda}(\lambda - A)^{-1}\boldsymbol{B}\, \mathrm{d}\lambda + \sum_n P_n(\boldsymbol{B})\mathrm{e}^{\lambda_n t} \tag{20}$$

where P_n represent (not necessarily orthogonal) projections upon the spaces generated by the eigenfunctions associated with eigenvalues λ_n, with $\mathrm{Re}\, \lambda_n > -k$ [13]. The integral on the left is bounded by some constant times e^{-kt}: taking k large enough, we see that this term does not contribute to the maximum of $\log \|T(t)\boldsymbol{B}\|/t$. Hence the supremum at (17) may be restricted to a finite-dimensional subspace of $D(A)$, where $\|\boldsymbol{B}\| = 1$ certainly implies $\|\boldsymbol{B}\|+\|A\boldsymbol{B}\| \leqslant M$ for some M. We have therefore found that

$$\Gamma = \sup_{(\boldsymbol{B}\in D(A), \|\boldsymbol{B}\|=1)} \sup_{(v\in C(\boldsymbol{B}))} \{\mathrm{Re}\, \langle v, A\boldsymbol{B}\rangle\} = \sup\{\mathrm{Re}\, \lambda : \lambda \in W(A)\}. \tag{21}$$

3. The minimal uniform growth rate

As stated before,

$$\gamma = \inf_{t>0} \frac{\log \|T(t)\|}{t} = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t}. \tag{22}$$

Since the limit exists (although it could be $-\infty$),

$$\gamma = \lim_{n\to\infty} \frac{\log \|T(n)\|}{n} = \lim_{n\to\infty} \log(\|T(n)\|^{1/n}) = \lim_{n\to\infty} \log(\|T(1)^n\|^{1/n}). \tag{23}$$

It is well known that

$$\lim_{n\to\infty} \|T(1)^n\|^{1/n}$$

is the *spectral radius* of the operator $T(1)$. Its relation to the spectrum of A is well studied (see, e.g., [1, 7]). In general, the point spectrum σ_p satisfies

$$\mathrm{e}^{t\sigma_p(A)} \subset \sigma_p(T(t)) \subset \mathrm{e}^{t\sigma_p(A)} \cup \{0\}. \tag{24}$$

Since $T(1)$ is compact and A has a purely discrete spectrum,

$$\sigma(T(1)) = \mathrm{e}^{\sigma(A)} \cup \{0\} \tag{25}$$

and by taking logarithms,

$$\gamma = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}. \tag{26}$$

Therefore, the minimal uniform growth rate is precisely the maximal growth rate. This does not follow in any obvious way from the original definition, although after showing that γ corresponds to the limit when $t \to \infty$ it is reasonable that a common bound for all initial conditions could be the same as the highest bound for every initial condition separately; that this fact is really not trivial is apparent from our use of the strong result that A has a pure point spectrum accumulating at $-\infty$.

One of the strongest points of this and the previous result is its independence of the norm, provided the conditions stated in the introduction are satisfied. This happens for a wide variety of L^p, Sobolev and Hölder norms. Of these, the L^1, L^2 and supremum norms are routinely used in dynamo studies. Of course, both results are equally true for any dissipative operator with the characteristics of A, but the growth rates are particularly meaningful for the dynamo problem.

4. Consequences

As usual, the space $X = L^2(\Omega)$ is the easiest to study. Ω will represent a smooth bounded domain of boundary $\partial\Omega$; for an operator A in $L^2(\Omega)$ its adjoint will be as usual denoted by A^*, and for the matrix $\nabla \boldsymbol{u}$, $(\nabla \boldsymbol{u})^*$ will again represent its adjoint. The complex conjugate is denoted by an overbar. As stated before, in this case the numerical range is

$$W(A) = \{(A\boldsymbol{B}, \boldsymbol{B}) : \|\boldsymbol{B}\| = 1\}. \tag{27}$$

If we assume that the flow is incompressible ($\nabla \cdot \boldsymbol{u} = 0$; $\boldsymbol{u}$ is of course assumed real), by standard applications of Gauss' theorem we find

$$\begin{aligned}
2\mathrm{Re}\,(A\boldsymbol{B}, \boldsymbol{B}) &= (A\boldsymbol{B}, \boldsymbol{B}) + (\boldsymbol{B}, A\boldsymbol{B}) = ((A + A^*)\boldsymbol{B}, \boldsymbol{B}) \\
&= 2\mathrm{Re} \iint_\Omega (\eta\Delta\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\boldsymbol{u}) \cdot \overline{\boldsymbol{B}}\,\mathrm{d}V \\
&= \int_{\partial\Omega} \eta \frac{\partial B^2}{\partial n}\,\mathrm{d}\sigma - 2\iint_\Omega \eta |\nabla\boldsymbol{B}|^2\,\mathrm{d}V + \int_{\partial\Omega} B^2 \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\sigma \\
&\quad + \iint_\Omega \boldsymbol{B}\cdot(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^*)\cdot\overline{\boldsymbol{B}}\,\mathrm{d}V.
\end{aligned} \tag{28}$$

Thus the value of Γ may be found by analysing the maximum of this amount when $\|\boldsymbol{B}\| = 1$. Although this could be difficult in general, if $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ (the fluid does not cross the boundary) and

$$\int_{\partial\Omega} \frac{\partial B^2}{\partial n}\, \mathrm{d}\sigma \leqslant 0$$

(there is no input of magnetic energy from the outside: this happens for all the usual homogeneous boundary conditions [14]) then we find the classical bound

$$\Gamma \leqslant \tfrac{1}{2}\|\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^*\|_\infty. \tag{29}$$

This result also follows from classical energy inequalities [14]; it is enough to find $\partial B^2/\partial t$ through the induction equation and show

$$\frac{\partial B^2}{\partial t} \leqslant M B^2 \tag{30}$$

where M is the supremum in (28), which proves

$$\|\boldsymbol{B}(t)\| \leqslant \|\boldsymbol{B}_0\| \mathrm{e}^{\frac{1}{2}Mt}. \tag{31}$$

Hence $\Gamma \leqslant M/2$; however, our result is finer in the sense that we prove that Γ is precisely $M/2$, i.e., the energy bound is optimal.

Another consequence of the expression of Γ in terms of the numerical range is the following: if $\boldsymbol{u}$ is a quasistatic solution of the Navier–Stokes equations (as it should be in a static kinematic dynamo problem) and $\boldsymbol{v}$ is a small perturbation of $\boldsymbol{u}$, $\boldsymbol{v}$ approximately satisfies the linearized Navier–Stokes equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nu\Delta\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \nabla p_1 \tag{32}$$

where ν stands for the viscosity and p_1 the perturbed pressure. The same arguments may be applied to this operator $S : \boldsymbol{v} \to \nu\Delta\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v}$, and therefore the same conclusions concerning the minimal and maximal uniform growth rates of $\boldsymbol{v}$. Again, the validity of equation (32) as a predicting instrument is limited in time: in this case because as soon as $\boldsymbol{v}$ reaches a comparable size to $\boldsymbol{u}$, the linear approximation breaks down. However, it is worth noting that all the terms $-\boldsymbol{u}\cdot\nabla v^2$, $-\nabla p_1\cdot\boldsymbol{v}$, $\boldsymbol{u}\cdot\nabla B^2$ cancel out when integrating in Ω, provided $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{n} = \nabla\cdot\boldsymbol{v} = 0$. In other words, the numerical ranges in $L^2(\Omega)$ coincide for A and S if $\nu = \eta$, and therefore the maximal uniform growth rates are the same. We may therefore assert that both magnetic field and velocity grow initially at the same maximum rate provided the dissipative parameters coincide; in other words, the convective effects, although essentially different for both magnitudes, increase them at the same maximum rate. The possibility that $\nu = \eta$ may be a rare coincidence, but it has been considered as a model in studies of MHD turbulence [15].

Finally, we may exploit the identity of γ and σ and a well known result [16] for the space $L^1(\Omega)$ to prove that if the minimum uniform growth rate is larger than a positive constant for all $\eta > 0$ and a $\mathcal{C}^\infty$ velocity field $\boldsymbol{u}$, then the flow induced by $\boldsymbol{u}$ has a positive topological entropy, and it is therefore chaotic.

5. Conclusions

We have studied the maximal and minimal uniform growth rates for a kinematic dynamo with a steady velocity field and for all time. The first one turns out to be the largest real part of all the complex numbers within the numerical range of the induction operator, and the second one the largest real part of all the eigenvalues of this operator. This is valid for all the usual norms; in the case of the quadratic mean norm (i.e. the magnetic energy) the maximal uniform growth rate coincides with the classical bound obtained by energy inequalities, which means that this is an optimal bound. One consequence is that the perturbed magnetic field and velocity initially grow at the same maximum rate.

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